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Knowledge and activities to enhance mental calculation

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ABSTRACT. Mental calculus is essential in the development of strategies for the solution of the four operations, fundamental to Mathematics. Several authors stand out for presenting mental calculation strategies, such as Backheuser (1933), Albuquerque (1951), Beishuizen (1993), Thompson (1999a), Parra (1996), Threlfall (2002), Boaler (2020). Through bibliographical research, we analyzed the strategies suggested by these authors and we have noticed that some authors name them, while others remain on examples, both with the intention of leading the reader to understand them. However, presenting a strategy and naming it is not enough for the individual to be able to understand and use it to be able to teach later. It is essential to recognize what knowledge is needed to trigger a strategy. In this article, we propose to analyze the mental calculation strategies described by these authors and list the knowledge needed for each of them, to understand what is necessary to know to formulate a strategy. From the analysis, we list four groups of essential knowledge for the development of mental calculation: Basic facts, Decomposition, Doubles and Network of Numerical relations of 10. In addition, we present activity suggestions that can be expanded and applied in the classroom, in order to help teachers to build knowledge and strategies with their students, enabling the development of mental calculus. Our research indicates that, before presenting a strategy, it is necessary to work on the knowledge that underlies it, otherwise it becomes difficult to perform mental calculations.

Keywords: addition; mental calculation; strategies; knowledge; activities.

Conhecimentos e atividades para potencializar o cálculo mental

RESUMO. O cálculo mental é essencial no desenvolvimento de estratégias para a solução das quatro operações, fundamentais para a Matemática. Diversos autores destacam-se por apresentar estratégias de cálculo mental, como Backheuser (1933), Albuquerque (1951), Beishuizen(1993), Thompson (1999a), Parra (1996), Threlfall (2002), Boaler (2020). Por meio de uma pesquisa bibliográfica, analisamos as estratégias sugeridas por estes autores e notamos que alguns as denominam, enquanto outros detém-se a exemplos, ambos com intenção de levar o leitor a compreendê-las. Porém, apresentar uma estratégia e nominá-la não é suficiente para que o indivíduo possa compreendê-la e utilizá-la para, posteriormente, poder ensinar. É imprescindível reconhecer quais os conhecimentos necessários para acionar uma estratégia. Neste artigo nos propomos a analisar as estratégias de cálculo mental descritas por estes autores e elencar os conhecimentos necessários a cada uma delas, de modo a compreender o que é preciso saber para formular uma estratégia. A partir da análise, elencamos quatro grupos de conhecimentos essenciais para o desenvolvimento do cálculo mental: Fatos básicos, Decomposição, Dobros e Rede de relações numéricas do 10. Para além disso, apresentamos sugestões de atividades que podem ser ampliadas e aplicadas em sala de aula, a fim de auxiliar professores a construírem com seus alunos conhecimentos e estratégias, possibilitando o desenvolvimento do cálculo mental. Nossa pesquisa indica que, antes de apresentar uma estratégia, é necessário trabalhar os conhecimentos que embasam a mesma, caso contrário, torna-se difícil realizar cálculos mentais.

Palavras-chave: adição; cálculo mental; estratégias; conhecimentos, atividades.

Conocimientos y actividades para mejorar el cálculo mental

RESUMEN. El cálculo mental es fundamental en el desarrollo de estrategias para la solución de las cuatro operaciones, fundamentales para las Matemáticas. Varios autores se destacan por presentar estrategias de cálculo mental, como Backheuser (1933), Albuquerque (1951), Beishuizen (1993), Thompson (1999a), Parra (1996), Threlfall (2002), Boaler (2020). A través de una investigación bibliográfica, analizamos las estrategias sugeridas por estos autores y notamos que algunos las nombran, mientras que otros se detienen en ejemplos, ambas con la intención de llevar al lector a comprenderlas. Sin embargo, presentar una estrategia y nombrarla no es suficiente para que el individuo sea capaz de comprenderla y utilizarla para luego poder enseñar. Es fundamental reconocer qué conocimientos se necesitan para desencadenar una estrategia. En este artículo nos proponemos analizar las estrategias de cálculo mental descritas por estos autores y enumerar los conocimientos necesarios para cada una de ellas, con el fin de comprender qué es necesario saber para formular una estrategia. A partir del análisis, enumeramos cuatro grupos de conocimientos esenciales para el desarrollo del cálculo mental: Operaciones básicas, Descomposición, Dobles y Red de relaciones numéricas de 10. Además, presentamos sugerencias de actividades que pueden ser ampliadas y aplicadas en el aula. con el fin de ayudar a los docentes a construir con sus alumnos conocimientos y estrategias, posibilitando el desarrollo del cálculo mental. Nuestra investigación indica que, antes de presentar una estrategia, es necesario trabajar en el conocimiento que la sustenta, alcontrario, se vuelve difícil realizar cálculos mentales.

Palabras-clave: adición; cálculo mental; estrategias; conocimiento; actividades.

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Introduction

Mental calculation is present in many everyday situations, such as calculating the change for a purchase at the supermarket, or when calculating the percentage of a discount into a given purchase, or even when calculating the best way to make an investment or financial application. In addition, it is considered that mental calculation is important for the learning of Mathematics, given that it allows the development of flexibility with numbers and numerical sense.

It is very common to associate mental calculation with quick and exact answers, through memorized answers. However, mental calculation is associated with choosing the best strategy to solve an operation, allowing approximations and estimates.

Mental calculus is pointed out by many researchers as an alternative to assist the teaching and learning of Mathematics. According to them, mental calculation is a way of promoting numerical sense, as it encourages the search for easier process based on the properties of numbers and operations (Reys, 1984; Parra, 1996; Abrantes, Serrazina, & Oliveira, 1999; Maclellan, 2001) and, at the same time, number sense is highly related to mental calculation skills (Markovits&Sowder, 1994). According to Thompson (1999b), we should teach mental calculation methodically because:

> Most calculations in real life are done in the head rather than on paper; Mental calculation promotes creative and independent thinking; It contributes to the development of better problem-solving skills; it develops sound number sense; It is a basis for developing estimation skill (Thompson, 1999b, p. 179).

The benefits of mental calculation described by these authors and others indicate that mental calculation is a way to enhance the learning of mathematical content, especially arithmetic, the basis of all Mathematics. We understand mental calculation as

> [...] those exact or approximate, which are carried out mentally, or with notes to support reasoning, which do not depend exclusively on the use of algorithms and counting. They are those who use strategies, numerical logical reasoning, which derive results from others memorized and have their actions validated by numerical and operational properties (Zancan & Sauerwein, 2017, p. 311).

It is observed that mental calculation is developed through strategies, which are historically debated and presented by different authors in pedagogical manuals such as Backheuser (1933) and Albuquerque (1951), in research reports or in scientific articles such as Beishuizen (1993), Thompson (1999a), Threlfall (2002), Boaler (2020) and Parra (1996). The definition of mental calculus itself is not a consensus among different authors. For example, each of the authors Parra (1996), Buys (2008), Reys (1984), Zancan and Sauerwein (2017) provide a different definition for mental calculation. However, there is a consensus on the relevance of mental calculation in the teaching of operations and the benefit for the teaching of the Mathematics.

When observing the indications of strategies for teaching mental calculation, some questions arise: 'How to teach these mental calculation strategies?', or 'What knowledge is needed for mental calculation strategies?', or even, 'What kind of activities can enhance mental calculation?'.

In this article, we propose to analyze the mental calculation strategies described by these authors and list the knowledge needed for each of them, to understand what is necessary to know to formulate a strategy. In addition, we present activity suggestions that can be expanded, deepened and implemented in the school routine, enabling the construction of knowledge identified as basic for mental calculation.

Mental calculation strategies in the bibliography

In this study, we carried out, through bibliographic research, a theoretical investigation to know, understand and analyze the mental calculation strategies mentioned by different authors. In this paper we will present the authors that stand out in the study of mental calculation strategies. The bibliographic research, which is considered the first step of all scientific research (Marconi & Lakatos, 2020) helped us to find the most common mental calculation strategies. The research used secondary sources, especially manuals, books, and articles on mental calculation.

At the same time, we carried out a study on the recommendation of mental calculation in the National Common Curricular Base (BNCC), a normative document that defines the set of essential learning that all students must develop throughout the stages and modalities of Basic Education. In the document, we find ten specific competences of Mathematics, where one of them describes that the student must be able to "Develop logical reasoning, the spirit of investigation and the ability to produce convincing arguments, using mathematical knowledge to understand and act in the world" (Brasil, 2018, p. 269). As well as detailed guidelines of the 'objects of knowledge' and 'skills' that must be developed in each 'thematic unit'. Thus, ensuring that all students have their learning and development rights assured, in accordance with the provisions of the National Education Plan (PNE).

The BNCC, related to Elementary School, has 247 skills, with an average of 27 items for the school year, covering the thematic units: Algebra (15%), Probability and Statistics (15%), Geometry (17%), Measurement Quantities (21%) and Number (32%). We noticed that nearly 1/3 of the thematic units are associated with the 'number'. An assessment of the skills, described in the thematic unit 'number', shows that they are based on reading, writing, counting, estimating, comparing, ordering quantities, composing and decomposing numbers, building basic facts related to the four operations, as well as solving and elaborating problems with numbers, from natural to real ones, involving the four operations, as well as building basic facts.

'Mental calculation' appears in 12 skills and 'calculation strategies' in six of them, most of them preceded by the word 'using' and followed by the word 'writing'. In other words, the BNCC regulates that students must develop skills related to numbers and their operations, suggesting, for this, the use of mental and written calculation or calculation strategies. However, at no point, it explicitly states that "mental calculation" should be taught. We realize that it permeates skills, always associated with writing.

In choosing the bibliographic sources for analysis, we prioritized materials that addressed mental calculation strategies because, as described by Manzo (1971, p. 72), bibliographic sources are "[...] means to define, solve, not only already known problems, but also to explore new areas, where the problems still have not been crystallized sufficiently". In Table 1 we present the selected and analyzed bibliographies, which versify, in their content, about mental calculation strategies for addition.

Table1. Consulted sources.

Obra	Autor	Ano
A Aritmética na escola nova	Everardo Backheuser	1933
Como se ensina aritmética.	Everardo Backheuser	1946
Metodologia da matemática	Irene de Albuquerque	1951
Metodologia do ensino primário	Afro do Amaral Fontoura	1961
Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades	Meindert Beishuizen	1993
Didática da matemática: reflexões psicopedagógicas	Cecília Parra	1996
Mental calculation strategies for addition and subtraction	Ian Thompson	1999a
Issues in teaching numeracy in primary schools	Ian Thompson	1999b
Flexible mental calculation	John Threlfall	2002
Estrategias cognitivas para el cálculo mental	Gálvez et al.	2011
Mentes sem barreiras: as chaves para destravar seu potencial ilimitado de aprendizagem.	Jo Boaler	2020

Source: The authors (2021).

As noted in Table 1, the sources consulted are heterogeneous, ranging from textbooks to scientific articles. We know that there are indications of mental calculation in the literature much earlier than 1933, but it was necessary to choose some sources, and this choice was made considering the type of mental calculation strategies described.

The analysis of the selected sources shows that some authors who mention mental calculation strategies in their writings do not name the strategies, such as Jo Boaler (2020) and Grecia Gálves et al. (2011). Other authors stand out for naming them. For example, Everardo Backheuser (1946) refers to a subtraction strategy called the Austrian Method, which consists of

> [...] completing the subtrahend until the minuend is equal. This method is currently considered as a teaching strategy for subtraction, which consists of transforming the subtractions into an addition to arrive at the answer. For example, 37 – 18 = __. The reasoning used is the following: 18 to 20 = 2 and 20 to 37 = 17, so the answer is 2+17 = 19 (Berticelli & Novaes, 2021, p. 727).

Meindert Beishuizen (1993) refers to two addition strategies as 1010 and N10. The 1010 strategy consists of adding the tens of both numbers first and then the units. For example, 35+23=(30+20)+(5+3)=50+8=58. While the N10 strategy consists of adding first the tens of the second ten and then the units. For example, 35+23=(35+20)+3=55+3=58.

Ian Thompson (1999a, p. 3) describes eight strategies for addition and subtraction, which he names as: "Doubles fact, Near-doubles, Subtraction as the invers of addition, Using fives, Bridging through ten, Compensation, Balancing". For the author, 'doubles fact' are those operations that involve doubles, for example, 18-9, 14-7. 'Near-doubles' are those with values close to double, for example, 7+8, calculated by doing 7+7+1. 'Subtraction as the inverse of addition' is about solving subtraction using memorized addition results, like knowing that 8-2 is 6, because 6+2=8. 'Using fives' is the strategy of decomposing numbers into 5+__ , for example, 6+8=5+1+5+2 =5+5+3=10+3. The 'bridge through 10' is the strategy of completing the nearest ten, for example, 8+5=8+2+3=10+3. 'Compensation' consists of thinking of a memorized result and compensating, for example, to solve 9+7, think that 10+7=17, so 9+7=16. Finally, 'balancing' is when we remove from one plot and add to the other, like thinking that 9+7=10+6. The author highlights that the most important strategies are: "[...] bridging through ten (up and down); partitioning single digit numbers; compensation (for adding or subtracting 9)" (Thompson, 1999a, p. 4).

The publication by John Threlfall (2002) is a literature review on mental calculation, where the author selects articles by researchers that deal with mental calculation strategies. Among them are mentioned Thompson (1999a) and Beishuizen (1993).

The analysis of mental calculation strategies for addition, indicated by the consulted authors, made it possible to establish categories and, thus, infer the necessary knowledge for each of these strategies. Like Severino (2016), we understand that in the process of analyzing the object we seek to transform what was composed and complex into something simple, that is, we analyze each strategy to understand it and then infer on which knowledge it is based. In other words, we use the process of "[...] decompose, dissect, interpret, study [...]" (Severino, 2016, p. 74) that is presented by Severino (2016), to analyze that strategy and understand the knowledge needed for them. We emphasize that there are mental calculation strategies for the four operations, however, for this text, we choose to analyze and bring strategies only for addition, as we understand that it is the basis for the other operations.

Subsequently, we evaluated this knowledge in the light of the skills identified in the BNCC and ended with activity suggestions to build this knowledge in practice, at any level of education.

Mental calculation strategies and knowledge for these strategies

In this section we will exemplify five mental calculation strategies for addition and infer the knowledge needed for them. These examples were stratified from the bibliography presented in Table 1.

Example 1: 8+6=8+2+4=10+4=14

To perform this addition, the strategy of completing 10, or Bridging through 10, described by Thompson (1999a) was activated. This strategy requires the following knowledge:

i. Knowing that 10 can be decomposed into 10=1+9=2+8=3+7=4+6=5+5= 6+4=7+3=8+2=9+1; and in this case, choose 8+2=10;

ii. Knowing that 6 can be decomposed into 6=1+5=2+4=3+3=4+2=5+1, and in this case, choose 6=2+4;

iii. Having understood the numerical compositions, in this case, that 10+4=14;

iv. Having numerical sense to know that you can think that from 8 to 10 there are 2 units missing, that is 6=2+4, which can exchange the operation 8+6 for the equivalent operation 8+2+4= 10+4.

From this example we can see that this strategy is built with knowledge of composition and numerical decomposition. The main compositions that must be memorized are those that result in 10 (exemplified in item i), together with the memories of addition results in which the number 10, or multiples of 10, is one of the parts (exemplified in item iii). This whole process is permeated by the numerical sense, understood as the ability to interact with numbers in a flexible and conceptual way (Boaler, 2018).

The same knowledge can be used for additions with numbers bigger than 10. For example, to calculate 27+16 we trigger that: 27+3=30 and that 16=3+13. Then the numerical sense makes it possible to change the expression 27+16, to 27+3+13 and to 30+13. To solve the addition 30+13, the decomposition of 13 as 10+3, is activated again, so, with numerical sense, 30+10+3=43. That is, all the knowledge mentioned above made it possible to replace 27+16 with 27+3+10+3, making it easier to obtain the result without the use of counting or the addition algorithm.

Example 2: 7+6=6+1+6=6+6+1=12+1=13

To perform this addition, the doubles fact strategy described by Thompson (1999a) was activated. This strategy requires the following knowledge:

i. Knowing that 6+6 =12;
ii. Knowing that 7 can be decomposed into 7=1+6=2+5=3+4=4+3=5+2=6+1;
iii. Having numerical sense to know that the equalities 6+7=6+(6+1)= (6+6)+1=12+1 are valid;
iv. Having memorized that 12+1=13.

From this example, we can see that this strategy is built on the memories of doubling numbers and numerical decomposition. Again, this whole process is permeated by numerical sense, which allows to obtain equivalent expressions.

This same example can be solved starting from the memory of the double of 7: 7+6=7+7-1=14-1=13. In this resolution we used the knowledge that 7+7=14 and that 6=7-1.

Example 3: 15+14=10+5+10+4=10+10+5+4=20+9=29

In this addition, the numerical decomposition strategy called 1010 by Beishuizen (1993) was activated. This strategy requires the following knowledge:

i. Knowing that 15 and 14 are made up of tens and ones, in the form 10+5 and 10+4, respectively;
ii. Having numerical sense to know that 15+14=(10+5)+(10+4)=10+10+5+4;
iii. Having memory of the result of 5+4=9;
iv. Having understood that numerical compositions, in this case, that 10+10=20 and that 20+9=29;

This example indicates that, for this strategy, knowledge of composition and decomposition into tens and units are necessary, associated with memories of some addition results and numerical sense.

Example 4: 16+13=16+10+3=26+3=29 or

16+13=13+16=13+10+6=23+6=29

In this addition, the strategy of decomposition into tens and units of one of the plots, called N10 by Beishuizen (1993) was activated. This strategy requires the following knowledge:

i. Knowing that 13=10+3 or 16=10+6;
ii. Knowing that 13+10=23 or 16+10=26;
iii. Having understood numerical compositions and knowing that 23+6=29 or 26+3=29.

Example 5: 26+19=26+20-1=46-1=45

In this addition, the strategy of rounding to the nearest tenth to one of the plots described by Thompson (1999a) was activated. This strategy requires the following knowledge:

i. Knowing that 26+20=46;
ii. Knowing that 19+1=20, and that 19=20-1;
iii. Having numerical sense to know that 26+19=26+20-1;

This example indicates that, for this strategy, memories of some addition results with multiples of 10 are needed, here 26+20. Also, knowing that 19=20-1, allowing you to use the 20-1 instead of 19.

The examples suggest that the same addition can be solved by different strategies. Like Threlfall (2002), we believe that the choice of strategy is related to the individual's prior knowledge, as mental calculation offers a "[...] model of flexible mental calculation that emphasizes personal knowledge as a determinant of how a solution path emerges in the context of particular calculations, by way of what is noticed by the individual about the numbers in the calculation" (Threlfall, 2002, p. 45).

Notice the strategies that can be used to obtain the solution of the addition 18+17=35:

Strategy 1: Completing 10 or one of its multiples (Example 1)

18+17=18+2+15=20+15 or18+17=17+18=17+3+15=20+15

Strategy 2: Using double memory (Example 2)

18+17=18+18-1=36-1 or 18+17=17+17+1=34+1

Strategy 3: Adding the units and then the tens of each number (Example 3)
18+17=10+10+8+7=20+15or18+17=8+7+10+10=15+20
Strategy 4: Decomposing one of the numbers into ten and one and adding (Example 4)
18+17=18+10+7=28+7 or18+17=18+7+10=25+10
Strategy 5: Rounding to the nearest ten (Example 5)
18+17=17+18=17+20-2=37-2

The analysis showed that these strategies are permeated by knowledge related to the memory of basic facts, decomposition of a number in additions or subtractions, memory of doubles and numerical properties. With this knowledge, it is easy to develop a mental calculation strategy.

Knowledge needed for additions by mental calculation

We identified that the necessary knowledge for mental calculation for additions can be gathered into four groups, which we call: Basic Facts, Decomposition, Doubles and Network of Numerical Relations of 10 (RRN of 10). Below, we describe our definition of this knowledge:

- Basic facts: These are addition or subtraction operations whose result does not exceed the nearest ten, that is, the sum is given only by changing the unit of the numbers. For example: 5+3; 7+2; 14+5; 21+8; 32+7.
- Decomposition: This fact consists in having memorized all the possible decompositions, in sums, of the numbers smaller than 10. Exemplified, all the combinations are shown below:

2=1+1;
3=1+2;
4=1+3=2+2;
5=1+4=2+3;
6=1+5=2+4=3+3;
7=1+6=2+5=3+4;
8=1+7=2+6=3+5=4+4;
9=1+8=2+7=3+6=4+5

- Doubles: Some strategies require the memorization of doubles, the most frequent are the doubles of the numbers from 1 to 20. As well as the invers operation. That is, knowing that 7+7=14 and that, consequently, 14-7=7. For example, 12+12=24; 24-12=12; 15+15=30; 30-15=15.
- Network of numerical relationships of 10 (RRN do 10): This group includes all the knowledge that involves 10 in the plots or in the result, as well as multiples of 10. Let's see some examples:

10=1+9=2+8=3+7=4+6=5+5;
10-1=9;10-2=8;10-7=3;...;10-9=1;
10+1=11;10+2=12;10+3=13;...;10+9=19;
10=11-1=12-2=13-3=14-4...;
20+1=21;20+2=22;20+3=23...;
20=21-1=22-2=23-3=24-4...
11+10=21;10+15=25; 30+14=44...;
13-10=3;25-10=15 ...

To trigger any mental calculation strategy, in addition to a minimum of mathematical knowledge, it is necessary to have numerical sense, that is, to have "[...] a mathematical mindset focused on making sense of numbers and quantities" (Boaler, 2020, p. 33). Numerical sense allows the interaction with numbers in a flexible and conceptual way (Boaler, 2020), which helps in the memorization of results. This flexibility with numbers allows the development of active mathematical thinking, where the student seeks to understand and make sense of the operations. With this, it is able to trigger mental calculations strategies, autonomously and safely, because "[...] the efficiency in a strategy lies in the thinking and understanding of each individual learner. No strategy will be enough for a student who still does not understand it" (Humphreys & Parker, 2019, p. 42).

In this way, by appropriating the knowledge described above, the individual can do mental calculation because, from them, together with numerical properties, they can think of strategies and obtain results of addition operations without using algorithms or counting. With the use of strategies, some results are memorized more easily, building new knowledge, which helps in the construction of new strategies. Thus begins a cyclical movement between knowledge and strategies, which improves numerical sense and flexibility with numbers. Through repetition, this movement generates memories.

On the other hand, an individual needs some basic knowledge, because, even if they are interested, they will not be able to build any mental calculation strategy. Without knowledge and without strategy, numerical sense and flexibility do not lead to the construction of new knowledge.

In addition, we understand that the teacher is an important and indispensable piece, as it is they who initiate the whole process. Once the decision has been made to insert mental calculation in the students' routine, the student starts the dynamics of knowledge construction, strategy construction and so on. Each new memory, each new knowledge, allows for new strategies, which results in more flexibility and number sense, and so, with time and practice, mental calculation develops.

The teacher, strongly intending to build mental calculation, needs planned activities that build the necessary knowledge, which were described above. In the next session we exemplify some activities that can be used to start the movement in search of mental calculation.

Suggestions of activities for the construction of mental calculation

From our experience with teacher training courses and the research we have carried out on this topic, we believe that the initial requirement for a successful teaching of mental calculation is the intention, that is to say, the teacher – one of the protagonists of the classroom (Berticelli & Zancan, 2021) – must be determined to insert mental calculation into their practice. The decision provokes a change of conception in relation to the teaching of Mathematics in the speech, in the way in which it will direct the students' thinking, encouraging them to seek different ways of solving, seeking to develop reasoning, stimulating the use of strategies and seeking to leave aside the use of algorithms in operations that can be solved by mental calculation strategies. In addition, students need to know that they are learning mental calculation, so the teacher needs to create moments of conversation about different ways of solving the same operation. For this, the teacher must be clear and make it clear to the students the reason for certain activities, as well as the knowledge on which they are built.

Regarding the activities to be proposed to the students, the first suggestion is to prioritize flexibility in terms of how the additions are presented. In general, an addition is presented in the form $a + b =$ ___, where the student has only one answer. In this sense, we propose that the same addition $a + b = c$ would be alternated among the twelve different possibilities, presented in Table 2:

Table 2. Different ways to present the addition.

a + b = __	a + __ = c	c = __ + b
b + a = __	__ + a = c	c = b + __
__ = a + b	b + __ = c	c = a + __
__ = b + a	___ + b = c	c = __ + a

Source: Prepared by the authors (2021).

This variation in the presentation of the operation allows the student to exercise flexibility with the numbers, so they become familiar with different presentations for the traditional 6+4=___; which can vary from ___=6+4 , 10=__+4 to 10=6+___. This approach strongly influences the understanding of algebra, when the student understands the equation 10=x+ 4 as equivalent to x+4=10.

Another suggestion is related to the degree of difficulty of the activities, which should be presented gradually, with activities grouped and named, allowing the student to understand how the thinking should proceed. In the example below we present extract of tasks, cut-outs, which must be varied and expanded to be used in the classroom.

In Figure 1, we exemplify the first tasks that must be grouped by knowledge, involving integers from 1 to 10.

Basic Facts	Decomposition	Doubles Memory	RRN of 10
1+7=	4=__+__	5+5=	4+6=
3+4=	4=__+__	6+6=	3+7=
7+2=	5=__+__	7+7=	2+8=
8+1=	5=__+__	8+8=	10=__+__
3+5=	7=__+__	9+9=	10=__+__
5+4=	7=__+__	6+6=	10=__+__
2+3=	7=__+__	8+8=	10=__+__

Figure 1. Example of activity.

Source: Prepared by the authors (2021).

The second activity block must contain numbers smaller than 20 in their operations, except for the decomposition block, which must remain in numbers smaller than 10, as we can see in Figure 2:

Basic Facts	Decomposition	Doubles Memory	RRN of 10 and its multiples
11+7=	5= 2+__	11+11=	14+6=
13+4=	7=__+ 3	12+12=	13+7=
17+2=	7= 2+__	13+13=	11+9=
18+1=	8=__+ 5	15+15=	18+10=
13+5=	9= 5+__	14= 7 +__	18+20=
15+4=	9= __+ 4	16= __+ 8	10+8=
12+3=	9= 3+__	18= 9+__	20+8=

Figure 2. Example of activity.

Source: Prepared by the authors (2021).

These tasks address solutions from 1 to 20 that we want students to memorize, which we provoke through repetition. Therefore, who determines how long they should be repeated are the students. Next, we explore how the addition is presented (Figure 3):

Basic Facts	Decomposition	Doubles Memory	RRN of 10
11+__=19	__+__= 4	15+15=	15=10+__
13+__=17	__+__= 4	16+16=	16=10+__
17+__=19	__+__ = 5	17+17=	10+__=18
___+3= 18	__+__= 5	18+18=	20+__=28
___+13=18	__+__= 6	32= __ +__	___+ 9=19
3+___=18	__+__= 6	34= __+ __	___+ 9=29
13+___=18	__+__= 6	36= __+__	___+19=29

Figure 3. Example of activity.

Source: Prepared by the authors (2021).

When the results of the additions grouped here by: Basic Facts, Decomposition, Memory of Doubles and the RRN of 10, for numbers less than 20, are memorized, the individual will be able to insert mental calculation strategies in his routine and start the dynamics of mental calculation. These strategies can be induced through activities designed for this purpose. Below we show examples of activities in which we name the strategies (Figure 4) and, with the teacher's encouragement, they can begin to be worked in the classroom. Only after memorizing the result for additions with numbers up to 20, we move on to larger numbers.

Bridging through ten	Memory of doubles	Decomposition into tens and ones
7+___=10 4=3+___ 7+4=7+3+1= 8+__=10 5=2+___ 8+5=8+2+3=	5+5= 5+6=5+5+1=10+1 = 7+7= 7+8=7+7+1=14+1 = 8+8= 8+9=8+8+1=16+1 =	14+10+2= 14+12= 15+10+3= 15+13= 28+10+1= 28+11=

Figure 4. Example of activity.

Source: Prepared by the authors (2021).

In Figures 5 and 6 we present activities with numbers smaller than 30 and smaller than 40, respectively, that explore mental calculation strategies. Is it important to notice that every time we want the student to activate a certain strategy, we need to present additions where the plots are constituted by numbers that facilitate or even induce the strategy that we want the student to use, since the numbers presented favor the choice of a certain strategy. Explaining in a clearer way, if the intention is to explore the bridge through 10, we must present situations that favor this choice, such as $8 + 5 =$ ___, instead of $8 + 8 =$ __.

In addition to these strategies, there are others that can be activated but they are not exemplified in this text. Here we address those that we consider the most important. We defend that, regardless of the level of education, the construction of knowledge should follow the sequence of the suggestions above. The time required for the construction of the knowledge may vary according to the level of education, the teacher's intention, the number of activities carried out in the school routine and the moments of conversation about the chosen thought.

Use the Bridging through ten	Use the Memory of doubles	Add tens and ones
8+5=	6+6=	14+10=
8+6=	6+7=	14+12=
8+7=	7+7=	15+10=
9+2=	7+8=	15+13=
9+3=	8+8=	25+10=
9+4=	8+9=	25+13=
9+5=	9+9=	25+15=

Figure 5. Example of activity.

Source: Prepared by the authors (2021).

Use the Bridging through ten	Use the Memory of doubles	Add tens and ones
18+5=	16+16=	24+10=
18+6=	16+17=	24+12=
18+7=	17+17=	25+10=
29+2=	17+18=	25+13=
29+3=	18+18=	35+10=
29+4=	18+19=	35+13=
29+5=	19+19=	35+15=

Figure 6. Example of activity.

Source: Prepared by the authors (2021).

We reinforce that, with the appropriation of this knowledge and memorization of these results, the individual may or may not build any mental calculation strategy. However, we are clear that, by carrying out the activities above, the individual is able to develop the necessary knowledge to formulate a mental calculation strategy. However, without this knowledge, no strategy of mental calculation, no matter how simple it is, can be built. We emphasize that these activities are intended to complement the other activities already developed in the classroom.

Final considerations

The possibility of various combinations of strategies makes each individual to choose the one that is more convenient for them, based on the sense and significance of the numbers established by them. The choice of different strategies is related to memories, making mental calculation something personal. Considering these memories, it is clear that it makes no sense to teach mental calculation strategies without first teaching them the necessary knowledge and having understood and memorized them, since without them, strategies do not happen. In other words, we understand that, in order to work with mental calculation strategies in the classroom, we must first focus on teaching the groups of knowledge listed above: 'basic facts', 'decomposition', 'doubles' and 'network of numerical relationships of 10', so that the individual learns, understands and memorizes them.

Many of the strategies described here can be found in Elementary School textbooks, inserted in examples without detailed explanations, where the author considers that the teacher has the ability to understand them. However, the reality of the classroom may not be this. When the teacher doesn't know, they stop exploring mental calculation and, with that, they are preventing the flexibility of thinking, which can be applied both to numbers and to any other situations.

With the necessary knowledge to choose addition strategies, the individual already has the basis built to develop the strategies of subtraction, multiplication and division, since addition is the primordial condition, considered the basis of all mental calculation. In this study, we defend that it is necessary, first, to teach the knowledge listed in this text, through activities prepared for this purpose. In a second moment, to motivate individuals to make use of mental calculation strategies. With the knowledge and the construction of the first strategies, the process starts and it feedbacks itself, that is, the more strategies are used, the more knowledge is built and memorized, with more knowledge, more strategies, and so on. With this, mental calculation is built and, with it, a better relationship with Mathematics.

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