ARTICLES

Initial conceptions about algebraic thinking of a group of mathematics teachers[1*]

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Abstract

This article, which, from a methodological point of view, resulted from a qualitative field research, aims to analyze the conceptions (ideas, representations or beliefs that participants have in relation to a given topic) of 36 teachers regarding algebraic thinking. Teachers participated in three synchronous online meetings, lasting three hours each, in which the aim was to discuss what algebraic thinking is and how to develop it. The aim was to identify, based on the answers given by participants to two questions that were part of the initial questionnaire of this research, which themes they associated with algebraic thinking. Theoretically, this research is supported by characteristic aspects of algebraic thinking advocated by different researchers in the field of algebraic education. Among the results, it could be highlighted that the teachers' conceptions are grouped into 13 categories, predominantly linked to the following actions related to the aforementioned way of thinking: treating unknown quantities as if they were known and performing calculations with them as we do with known values. Two actions of extreme relevance for the development and mobilization of algebraic thinking were not evidenced by participants in their conceptions: perceiving the relationships of variations and covariations; and understanding the different roles played by the equal sign.

Keywords

Algebraic thinking – Actions related to algebraic thinking – Teachers' conceptions – Themes linked to algebraic thinking.

1- **Data availability:** The entire dataset supporting the results of this study was published in the article itself or in: Doi:10.48331/scielodata.EMZ8VB (https://data.scielo.org/dataset.xhtml?persistentId=doi%3A10.48331%2Fscielodata.EMZ8VB&version=DRAFT)

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Introduction

This article is the first fruit of the research project: *An analysis of the knowledge of basic school teachers in relation to algebraic thinking*, funded by the Research Incentive Plan (PIPEq) – notice 11905/2022 – of the Pontifical Catholic University of São Paulo (PUC-SP). This research is part of the Algebraic Education Research Group (GPEA) and is linked to the project *Algebra in Basic Education*, developed by the group linked to the line of research *Mathematics in the curricular structure and teacher training*, one of the areas followed in the Graduate Studies Program in Mathematics Education at PUC-SP.

The project was developed through three synchronous online meetings held via the *Zoom* platform and lasting three hours each, with teachers of different ages and with different levels of teaching experience, who teach mathematics at different educational levels. These meetings aimed to discuss with participants what algebraic thinking is and how to develop it.

Although the main aim of the project was to identify the knowledge of teachers in relation to the aforementioned theme, we understand that, in order to achieve this, it is essential to understand the initial conceptions of these participants about algebraic thinking. According to Martins (2012), this is a relevant theme in mathematics education and, specifically, in research on mathematics didactics. An explanation for this relevance can be found in the study by Thompson (1997). According to the author:

> [...] there is a strong reason to believe that in Mathematics, teachers' conceptions (beliefs, views and preferences) about the content and its teaching play an important role in terms of their efficiency as primary mediators between the content and the students (Thompson, 1997, p. 12).

This article sought to analyze the conceptions of some teachers regarding algebraic thinking, in which we work with the answers of the 36 participants who filled out a questionnaire prepared by the authors and made available through an Office form to those who registered as interested in participating in the three meetings.

This article was structured as follows: introduction, methodological procedures used in the research, theoretical basis – especially focusing on the idea of conception and the conceptualization of algebraic thinking and the main elements that characterize it – analysis and discussion of data and, finally, the conclusions that we were able to reach from the study carried out.

Methodological procedures used in the research to obtain data

From a methodological point of view, the project included a qualitative field research to obtain data from teachers interested in deepening their understanding of the constituent elements of algebraic thinking and developing this way of thinking in their teaching practices. To begin the research, we announced on social media that three free, synchronous online meetings would be held for teachers who teach mathematics and were interested in discussing what algebraic thinking is and how to develop it. We received 116 registrations from all regions of the country and even from abroad. We then sent to all those who registered, via an Office form, a questionnaire entitled *The view of teachers who teach mathematics regarding algebraic thinking.* This instrument contained 19 questions divided into two dimensions: (i) brief characterization of teachers; and (ii) their conceptions regarding algebraic thinking. Specifically regarding the second dimension, the questions were as follows:

- What do you understand by the term algebraic thinking? Explain in as much detail as you can.
- In the disciplines included during your undergraduate, graduate or other courses, was there any specific approach to topics related to algebraic thinking? Explain.
- According to your understanding, which topics would be involved in studies on algebraic thinking?
- Do you consider the development of algebraic thinking in basic education to be important? Justify your answer.
- What knowledge do you have about the inclusion of algebraic thinking in the documents that guide Brazilian education, at the federal, state and municipal levels?

The three meetings held later on the Zoom platform were recorded in audio and video. In the first meeting, participants interacted with proponents by also answering, via the Mentimeter tool, a series of proposed questions, namely: (i) Are thinking and reasoning synonymous? (ii) Write five words that, in your opinion, are related to mathematical thinking. (iii) Write five words that, in your opinion, are related to algebra. (iv) Write five words that, in your opinion, are related to algebraic thinking. (v) When teaching algebra, consequently, is the students' algebraic thinking being developed? (vi) Is there a transition from arithmetic to algebra and, consequently, should the teaching of arithmetic precede that of algebra?

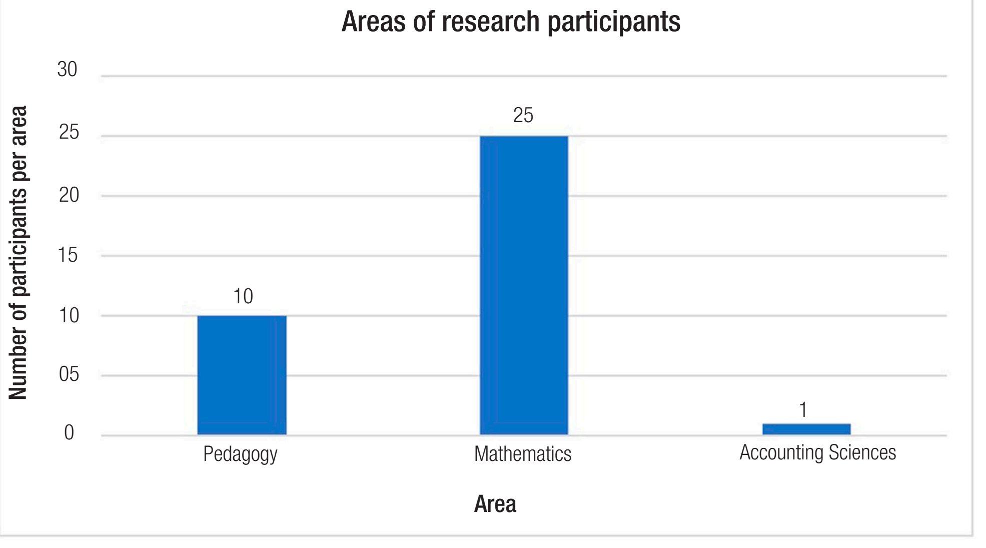
In the second meeting, participants developed, based on skills from the BNCC – National Common Curricular Base - three activities aimed at developing, by the student, some aspects of algebraic thinking. Finally, in the third meeting, they presented and discussed the productions of groups carried out in the previous meeting and, then, via Google Forms, they answered two final questionnaires. The first of them dealt with their conceptions of algebraic thinking after participating in the workshops and in that instrument, we asked them to detail, as much as they could, what they understood by algebraic thinking. The second consisted of an assessment of the meetings in which they participated.

In this article, some information about participants (age, length of teaching experience, and educational background) was specifically presented, as well as an analysis of their conceptions regarding algebraic thinking based on their answers to two of the questions in the first questionnaire, namely: (a) What do you understand by the term algebraic thought? Explain in as much detail as you can; and (b) According to your understanding, which topics would be involved in studies on algebraic thinking?

Brief characterization of research participants

As already mentioned, 36 responses to the questionnaire containing the two questions analyzed in this article were obtained. Participants' educational background (completed or in progress) are distributed as shown in Graph 1.

Graph 1 – Initial educational areas of research participants



Source: Research data

Of this group of 25 graduates or undergraduates in mathematics, 10 are in pedagogy and 1 in accounting sciences, as shown in Table 1, 4 participants are still completing their initial undergraduate course; 6 are graduates, but are not taking any course after their first degree, 1 took a short course, 12 are specialists, 6 have master's degree and 7 have doctorate degree, indicating that the meetings held during the research project attracted a very heterogeneous population.

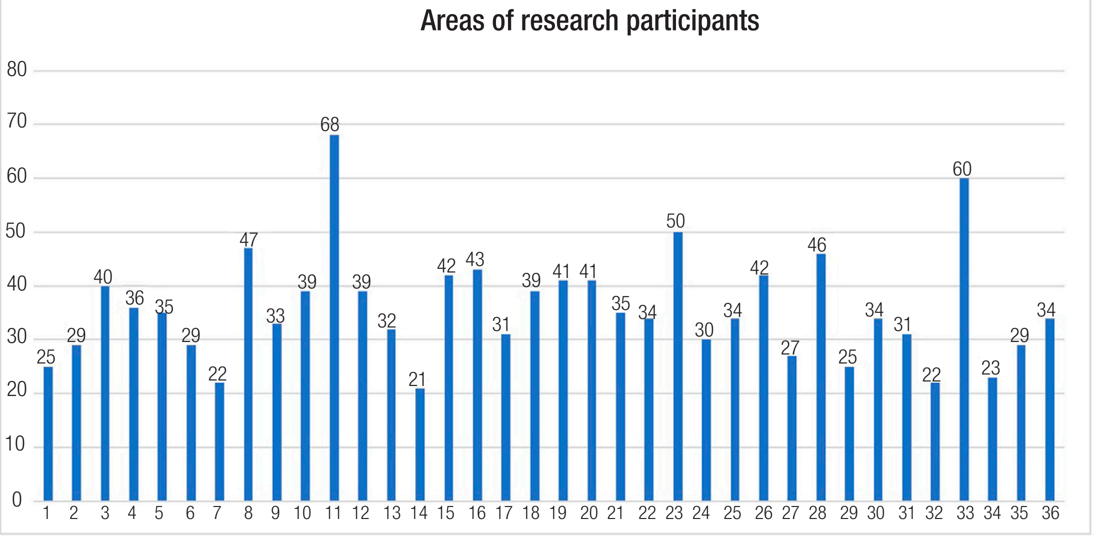
Table 1 – Maximum qualification of research participants

Type of course (maximum qualification)	Number of participants
PhD	7
Master's	6
Specialization	12
Short courses	1
Graduate	6
Undergraduate studies still in progress	4
TOTAL	36

Source: Research data.

The ages of participants are shown in the Graph 2.

Graph 2 – Ages of questionnaire respondents



Source: Research data

From Graph 2, it could be inferred that the average age of research participants is 37 years. Regarding their teaching experience, these are distributed in relation to the classes listed in Table 2.

Table 2 – Participants' teaching experience

Teaching experience	Number of participants	Relative frequency
0 to 6 years and 11 months	14	37.84%
7 years to 13 years and 11 months	13	35.14%
14 years to 20 years and 11 months	4	10.81%
21 years to 27 years and 11 months	3	8.11%
28 years to 34 years and 11 months	0	0.00%
35 years to 42 years	2	5.41%
Total	36	100%

Source: Research data.

It could be observed from Table 2 that 37.84% of participants have maximum of 6 years and 11 months of teaching experience; 35.14% have 13 years and 11 months of maximum teaching experience and, consequently, 72.97% of participants have been teaching for a maximum of 13 years and 11 months. It was also observed that 18.92% of participants have between 14 years and 34 years and 11 months of teaching experience. Only 5.51% of participants have been teaching for more than 35 years.

It is important to highlight that although this article does not establish relationships between the profiles of participants and their conceptions about algebraic thinking, we consider it relevant to present the brief characterization of teachers participating in the research, even without considering this information in the analyses, so that readers can have a minimum of clarity about who these participants are and how heterogeneous the group is. There are, evidently, possible paths to be followed in future investigations linking the conceptions of teachers to their profiles. Among other perspectives, for example, the conceptions of algebraic thinking expressed by pedagogical teachers could be compared to those expressed by graduates in mathematics.

Methodological procedures

To identify the conceptions of research participants regarding algebraic thinking, we initially compiled, using an Excel spreadsheet automatically generated by the Office forms tool, their responses to the previously mentioned questionnaire. By carefully analyzing how each subject responded to the question “What do you understand by the term algebraic thinking? Explain in as much detail as you can”, we first inferred the conception expressed by each subject and then identified convergences and divergences in the conceptions of the different participants that allowed us to rewrite them and group them into 13 more general categories that, in our view, provide a better overview of what participants thought about the topic than working with 36 different conceptions.

Once the 13 conceptions had been identified, they were analyzed using theoretical considerations from different authors about algebraic thinking and its characteristic elements, as we will explain later in this article. But first, it is necessary to clarify what it is understand by *conception* in this article.

What we mean by the term conception

As indicated by different studies in the field of education and, especially, mathematics education – such as, for example, Artigue (1989), Balacheff (1995), Almouloud (2007), Lima and Silva Neto (2012) and Martins (2012) – the term conception is polysemic and used in different studies often without a precise explanation of the meaning attributed to it. In this regard, Lima (2009) states that, in some cases, the term is used as a synonym for spontaneous reasoning, alternative structure, belief, representations, among others. Corroborating this idea, Thompson (1992) defines conception as a “general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences and the like” (Thompson, 1992 *apud* Opre, 2015, p. 230).

In this article, we assume the meaning presented by Lima (2009, p. 29):

> [...] a conception can be understood as an idea, a representation or a belief that a subject has about something, [...] as a specific type of individual knowledge constructed in the subject’s interaction with the environment.

According to Silva Neto (2012), based on the ideas of the aforementioned author, when analyzing teachers' conceptions about a given mathematical topic, it is necessary to consider that these:

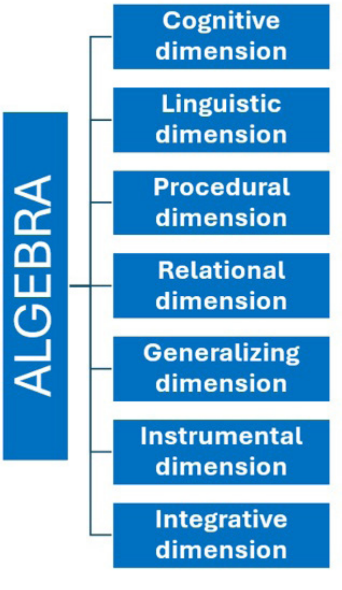
> [...] are formed from the teacher's experiences (life experiences, work experiences and background), that is, they are influenced by the context in which the teacher lives and/or develops his/her work. On the other hand, these conceptions also influence this context essentially in the practical performance of the teaching activities (Silva Neto, 2012, p. 33).

Based on the considerations presented in this section, we could be more explicitly define the aim of this article: to identify the ideas, representations or beliefs expressed about algebraic thinking by a group of 36 teachers who teach mathematics. That is, the individual knowledge that they have constructed regarding the aforementioned topic based on their experiences in different areas, such as background, their professional activities, their readings, etc. However, in order to understand the conceptions of a group of teachers regarding algebraic thinking, we first need to explain what characterizes this specific way of thinking. This is the task to which we dedicate ourselves in the following section.

Characteristic aspects of algebraic thinking

Based on different understandings by different authors about what characterizes algebra, Lima; Bianchini and Lima (2023) propose seven non-hierarchical, complementary and articulated dimensions that, when together, define this area of mathematics, as indicated in Figure 1.

Figure 1– Dimensions that constitute algebra



Source: Authors, based on the ideas of Lima; Bianchini and Lima (2023).

Regarding the different dimensions that constitute algebra mentioned in Figure 1, linguistics refers to algebra as a specific language, with syntax, semantics and pragmatics, for the solution of problems. The procedural dimension is associated with the performance of operations with abstract entities that are represented by symbols of universal scope. The relational dimension is linked to the study of relationships between two or more quantities, encompassing their perception, description, representation and manipulation, as well as the generation of models. The generalizing dimension is directly related to generalized computational processes and makes it possible, through the use of symbols, to perform, in a generic way, all operations involved in arithmetic. Regarding the instrumental dimension, it could be said that it is what translates the essence of algebra as a tool for different areas of knowledge, grouping different languages for modeling situations in mathematical and extra-mathematical contexts. Regarding the integrative dimension:

> [...] it encompasses the general structures common to all parts of Mathematics, which is present in practically every type of mathematical knowledge. It is related to what is true in all other branches of this science (Lima; Bianchini; Lima, 2023, p. 84-85).

Algebraic thinking is precisely linked to the dimension not yet mentioned in the previous paragraph: the cognitive dimension, which according to the aforementioned authors, is composed of:

> [...] clear paths of thought (including reverse thinking), interpretation and understanding of everyday situations, that is, essentially algebraic modes of thought, aiming at action, doing or knowing. It constitutes a powerful cognitive tool that contributes to the recovery of ideas and concepts for the establishment and understanding of relationships, generalizations, verifications, analyses, syntheses and abstractions in general (Lima; Bianchini; Lima, 2023, p. 83-84).

Analyzing the different ways of defining algebraic thinking used by the main authors of reference in the area of algebraic education, such as: Argentinean Abrahan Arcavi, the Brazilians Analúcia Schliemann, Angela Marta Pereira das Dores Savioli, Daniele Peres da Silva, Jadilson Ramos Almeida, Marcelo Câmara Santos, Rômulo Lins and Terezinha Nunes Carraher, Canadian Carolyn Kieran, Americans Barbara Brizuela, Maria Blanton and James Kaput, Guatemalan Luis Radford, Englishman David William Carraher and Mexican Luis Moreno, some actions related to algebraic thinking and its consequent development from the earliest ages were identified and summarize in Figure 2.

Figure 2 – Actions related to algebraic thinking

Algebraic thinking is related to the following actions:

*Analyzing relationships between quantities, detecting structure, studying change, conjecturing, generalizing, modeling, justifying, proving and predicting;

*Perceiving the relationships of variations and covariations;

*Using a variety of representations (related to gestural, natural, pictorial and symbolic languages) that allow dealing with quantitative situations in a relational way;

*Producing meaning for algebra and algebraic objects.

*Perceiving regularities in arithmetic operations;

*Differentiating the uses of variables (unknown, functional relation, generic number and parameter);

*Understanding the different roles played by the equal sign (operator, indicator of equivalence);

*Treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic.

Source: Prepared by the authors.

As Kaput (1999, 2008) points out, algebraic thinking is an exclusively human activity that arises from established generalizations as a result of conjectures about data and mathematical relationships, as well as through an increasingly formal language, used in argumentation. It is not a way of thinking exclusive to algebra; it permeates all mathematics, since it arises as an expansion of reasoning that goes beyond particular cases. In this author's view, algebraic thinking is composed of three aspects: generalized arithmetic or quantitative thinking, functional thinking and modeling, as illustrated in Figure 3.

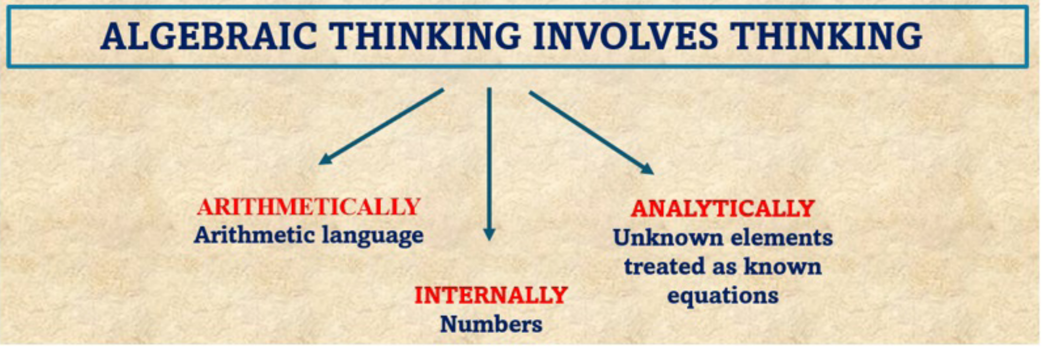
Figure 3 – Aspects of algebraic thinking according to James Kaput

Generalized arithmetic or quantitative thinking	Functional thinking	Modeling
• Generalization about arithmetic operations and their properties and the thought about the relationships between numbers.	• Generalization of numerical standards to describe functional relationships, in addition to perceiving the relationships between variables and (co)variables.	• Generalization of regularities in everyday situations in which regularity is secondary in relation to the most general aim of the task.

Source: Prepared by the authors based on the ideas of Kaput (2008).

Lins (1992) also conceives of algebraic thinking as being an amalgam of three different ways of thinking: arithmetically, internally and analytically, which is shown in Figure 4.

Figure 4 – Algebraic thinking aspects in the sense of Rômulo Lins



Source: Prepared by the authors based on the ideas of Lins (1992).

According to Lins (1992), arithmetically thinking is related to the use of arithmetic operations to represent and manipulate the relationships present in a given mathematical model. Internal thinking, on the other hand, concerns the distinction between:

> [..] internal solutions, that is, those that occur within the limits of the semantic field (way of producing meaning) of numbers and arithmetic operations, and not through the manipulation of non-arithmetic models (Lins, 1992, p. 14).

Finally, the analytical thinking mentioned in Figure 4 refers to the potential of this way of thinking as a "method for seeking the truth" and to the fact that, in algebraic thinking, the unknown is treated as known (Lins, 1992).

For the aforementioned author, an essential aspect related to algebraic thinking that can summarize what is most important about this way of thinking is the attribution, by the subject who develops and mobilizes it, of meaning to algebra and the different objects of this subarea of mathematics.

Another important aspect to be highlighted is the fact that, although symbols are at the core of algebraic thinking, it should not be reduced to algebraic transformism, but rather to the opposite, the most important thing is the construction of meaning, it is thinking with understanding, understanding what each symbol means (Kaput; Blanton; Moreno, 2008).

One of the actions linked to algebraic thinking is to differentiate the uses of variables. According to Ursini *et al.* (2005), authors of the theoretical model called 3 Uses of Variables (3UV), there are three main uses of variables in algebra used in basic education: unknown, generic number and functional relation[4]. Thus, it is essential that students, throughout their educational path, learn to differentiate, interpret, symbolize and manipulate variables in each of these uses and that they move between them naturally.

Identifying and adopting the different uses of variables is an action related to algebraic thinking that must be present in a teacher's conception of the characteristics of this way of thinking. This is because, if this knowledge is in the mathematics teacher's repertoire, their students will benefit, since the different uses can be explored in the classroom, in actions such as those indicated in Table 3.

Table 3 – Summary of the 3UV Model

Variable as an unknown
I1. Recognize and identify, in a problematic situation, the presence of something unknown that can be determined considering the constraints of the problem.
I2. Interpret the symbolic variable that appears in an equation, as the representation of specific values.
I3. Replace the variable with the value or values that make the equation a true statement.
I4. Determine the unknown quantity that appears in equations or problems, performing algebraic, arithmetic or both types of operations.
I5. Symbolize the unknown quantities identified in a specific situation and use them to formulate equations.
Variable as a generic number
G1. Recognize patterns and perceive rules and methods in sequences and families of problems.
G2. Interpret the symbolic variable as the representation of a general, undetermined entity that can assume any value.
G3. Deduce general rules and methods in sequences and families of problems.
G4. Manipulate (simplify, develop) the symbolic variable.
G5. Symbolize statements, rules or general methods.
Variable in a functional relationship
F1. Recognize the correspondence between related variables, regardless of the representation used (tables, graphs, verbal problems, analytical expressions).
F2. Determine the values of the dependent variable, given the values of the independent variable.
F3. Determine the values of the independent variable, given the values of the dependent variable.
F4. Recognize the joint variation of the variables involved in a functional relationship, regardless of the representation used (tables, graphs, verbal problems, analytical expressions).
F5. Determine the ranges of variation of one of the variables, given the range of variation of the other.
F6. Symbolize a functional relationship, based on the analysis of data from a problem.

Source: Ursini *et al*. (2005, p. 35-37) *apud* Beltrame (2009, p. 40-41).

4- In our perception, we could identify a fourth use of the variable: the variable as a parameter. However, the authors who developed the 3UV Model understand that this use is a particular case of the variable acting as a generic number.

Due to its importance for the development of algebraic thinking, the action indicated in Figure 2, regarding understanding the different roles played by the equal sign, also deserves to be highlighted. According to Lima; Bianchini and Lima (2023),

> [...] in an arithmetic operation [...] the equal sign is perceived as an "operator", and the result of the operation that occurs in the first member should be indicated after the sign. For example, in the expression 2 + 3 = 5, the equal sign represents the command "operate", and immediately after this sign, the result of the operation must be presented, that is, when adding 2 and 3, the sum is 5. In the algebraic field, the equal sign, in turn, needs to play the role of an indicator of equivalence, that is, what is in the first member of the equation is numerically equivalent to what is in the second member. Such understanding is essential for, for example, understanding the manipulation of quantities (including unknowns), whose statement would be: "if I have the same quantity in both members of the equality, it is possible to remove equal quantities from both members without altering the equality". The principle of manipulating quantities/unknowns is, then, what justifies the technique of transposing one member of the equality to the other, using the inverse operation (Lima; Bianchini; Lima, 2023, p. 91-92).

To summarize what, in essence, constitutes algebraic thinking, we will present some more considerations present in Lima; Bianchini and Lima (2023). In the authors' view,

> [...] thinking algebraically would ultimately mean bringing Algebra into oneself, into its cognitive structure, making it possible to use all the constituent elements of the algebraic field: discovering the unknown, representing one object from another (the number from the letter), establishing relationships that are not clearly stated in the presented situation, operating, generalizing, proposing models and patterns. Thus, Algebra is no longer merely understood as a field of knowledge defined in the curriculum; cognitively, it constitutes itself as a tool for thinking, for sophistication and amplification of the cognitive structure (Lima; Bianchini; Lima, 2023, p. 102-103).

Having explained the elements that characterize algebraic thinking, we can finally present the conceptions about this way of thinking expressed by teachers participating in this research. This is the aim of the following section.

What do the research participants conceive of as algebraic thinking?

As we detailed in the methodology section used in this article for the production and organization of data, after analyzing the answers of each of the 36 participants to the question: what do you understand by the term algebraic thinking? Explain in as much detail as you can. We first identified the conceptions of each teacher for the term algebraic thinking, then, explaining the convergences and divergences among them, we grouped them into 13 distinct concepts presented in Table 4.

Table 4 – Conceptions about algebraic thinking expressed by participants

Conception	
C1	Algebraic thinking is a tool, a path to solve problems
C2	Algebraic thinking is a language to solve problems
C3	Algebraic thinking is a tool to simplify problems
C4	Algebraic thinking is something associated with logical reasoning, but disconnected from real situations
C5	Algebraic thinking is a practical and realistic view of Mathematics
C6	Algebraic thinking is something specific to Algebra
C7	Algebraic thinking is something directly related to Algebra
C8	Algebraic thinking is a way of explaining the properties of Algebra
C9	Algebraic thinking is a simplification of the teaching of Algebra
C10	Algebraic thinking is a process related to the articulation of mathematical concepts and their everyday applications
C11	Algebraic thinking is a tool for formalizing spontaneous mathematical knowledge for scientific mathematical knowledge
C12	Algebraic thinking is directly related to numbers
C13	Algebraic thinking is a process of understanding that begins with numbers and progresses towards variables and/or unknowns

Source: Research data.

Conception C1 has a utilitarian character, being linked to procedural aspects and the mathematization of a problem. As we can observe by grouping together some answers given by participants who expressed this conception, algebraic thinking is related to a logically structured or standardized path to face, in a clear and logical way, different situations - including daily ones - in different areas of knowledge. They consider that this way of thinking goes beyond that provided by basic arithmetic and is fundamental for understanding and solving more complex problems in mathematics, in its different subareas, and in several other disciplines. It is a way of reasoning mathematically that can be represented by symbols, letters, rules and algebraic concepts. It can be considered as the ability to express in generic terms something that is variable, abstract or non-numerical.

Articulating conception C1 with the actions related to algebraic thinking shown in Figure 2, the following was predominantly observed: using a variety of representations that allow dealing with quantitative situations in a relational way and treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic.

Conception C2 is also utilitarian in nature, but it is related to linguistic aspects and not necessarily procedural as observed in C1. It is also linked to the process of mathematizing a problem. These impressions are confirmed by the grouping of some extracts of participants' answers, in which they state that: algebraic thinking is a language for translating information given in the native language into symbolic language, representing unknown quantities or variables in mathematical expressions and equations, thinking of numbers as unknowns, associating terms regardless of whether or not they are explicit in an operation or everyday situation, transforming other mathematical representations into algebraic writing and vice versa, and solving problems based on these procedures. They also emphasize that it is a way of thinking and language; a cognitive and mathematical ability related to knowledge and skills linked to variables, unknowns, relationships, hypotheses, reasoning, representations, arguments, presentations and exposition of ideas, generalization (the central idea in this way of thinking) with its own resources and not only through formal language, determination of patterns. It allows the individual to explain how he/she thought, argued and justified when solving a problem. Developing algebraic thinking means developing the cognitive ability that involves the creative capacity for abstraction in mathematical representations, with concrete or real application, to solve a mathematical problem. It is also linked to subjectivity, logic and intentionality.

The following actions linked to algebraic thinking are explicitly stated in conception C2: analyzing relationships between quantities, detecting structure, studying change, conjecturing, generalizing, modeling, justifying, proving and predicting; using a variety of representations (related to gestural, natural, pictorial and symbolic languages) that allow dealing with quantitative situations in a relational way; differentiating the uses of variables (unknown, functional relation, generic number and parameter); and treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic.

Conception C3 is also utilitarian, of a simplifying nature and linked to the idea of algebra as a generalization of arithmetic. In the words of participants who demonstrated this conception, algebraic thinking is an evolutionary response of mathematics to complex problems in which the use of arithmetic would be excessively extensive and laborious, being necessary to apply this response consciously, considering its logical structure, meanings and application domain. Regarding the actions related to algebraic thinking; in this conception, we perceive the explicit presence of the following: treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic. Implicitly, we can infer the presence of perceiving regularities in arithmetic operations.

Conception C4 is logicist, of a purely abstract nature and disconnected from real situations. According to the subject who expressed it, algebraic thinking is linked to logical reasoning and is uncomfortable for students because they do not establish relationships with the real world. This conception does not include any of the actions related to algebraic thinking listed in Figure 2. Likewise, conception C5 is also not linked to any of these actions. Furthermore, it has a nature opposite to that observed in C4, since it is practical and realistic in nature, and is associated, according to the subject who expressed it, with the ability to see and think about mathematics in a practical and realistic way.

Conceptions C6 and C7 have similar characteristics, but differ in one essential aspect: while C6 is limitedly algebraic, C7 is strongly related to algebra, but not limited to it. The answers of participants who expressed these conceptions ratify these ideas. Those who present conception C6 indicate that algebraic thinking is something exclusive to algebra and directly related to the manipulation of algebraic language and abstraction. Those with conception C7 affirm that algebraic thinking is related to the true meaning of mathematics – algebra – without which it is very difficult to teach it. In C6, we identified the following action related to algebraic thinking: treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic. In C7, there is no evidence of any of these actions linked to algebraic thinking.

Regarding conception C8, we can classify it as explanatory, since participants who expressed it state that algebraic thinking is a way of explaining algebraic properties with clarity and precision. It is directly linked to the action: producing meaning for algebra and algebraic objects.

Conception C9 can be understood as being of a pedagogical-simplifying nature, since the subject who presented it understands algebraic thinking as a simplified way of teaching algebra. It is not linked to any action related to algebraic thinking, as evidenced in Figure 2.

Conception C10 shows a unifying and applied character. Participants who stated that they have this conception understand it as a process, which begins in the early years of basic education and continues through higher education, related to the articulation of mathematical concepts and their applications in everyday life. This conception is not directly related to any action intrinsic to algebraic thinking indicated in Figure 2.

Conception C11 can be understood as having an instrumental-formalizing nature from spontaneous mathematical knowledge to scientific knowledge, exactly as described by participants who expressed it: a way of thinking that allows the formalization of everyday mathematical knowledge into scientific/school mathematical knowledge. At least explicitly, there is no mention, in this conception, of any action related to algebraic thinking. Likewise, conception C12, of an arithmetic nature, is also not linked to any of these actions. Participants explained it simply by mentioning that algebraic thinking is related to working with an emphasis on numbers.

Finally, conception C13 can be interpreted as being of a procedural nature: from numbers to variables. Participants explained it by stating that algebraic thinking is the ability to understand the equalities and inequalities between objects, numbers and subsequently the advancement to the understanding of variables and/or unknowns. This conception includes the following actions related to algebraic thinking: analyzing relationships between quantities, detecting structure, studying change, conjecturing, generalizing, modeling, justifying, proving and predicting; differentiating the uses of variables (unknown, functional relationship, generic number and parameter); treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic.

The associations that could be made between conceptions expressed by research participants and the actions inherent to algebraic thinking were summarized in Table 5.

Table 5 – Actions related to algebraic thinking present in the conceptions expressed by research participants

Action related to algebraic thinking	Explicitly present in the conceptions
Analyzing relationships between quantities, detecting structure, studying change, conjecturing, generalizing, modeling, justifying, proving and predicting	C2, C13
Perceiving the relationships of variations and covariations	not evident in any of the conceptions
Employing a variety of representations (related to gestural, natural, pictorial and symbolic languages) that allow dealing with quantitative situations in a relational way	C1, C2
Producing meaning for algebra and algebraic objects	C8
Perceiving regularities in arithmetic operations	C3
Differentiating the uses of variables (unknown, functional relation, generic number and parameter)	C2, C13
Understanding the different roles assumed by the equal sign (operator, indicator of equivalence)	not evident in any of the conceptions
Treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic	C1, C2, C3, C6, C13

Source: Research data.

From the analysis of Table 5, it was observed that the action related to algebraic thinking most explicitly linked to the conceptions of teachers is that of treating unknown quantities as if they were known and performing calculations with them as we do with known values in arithmetic. The predominance of this action can possibly be explained by the importance given in basic education and even in teacher training courses to the idea of algebra as a generalization of arithmetic, which reduces algebraic thinking according to Kaput (2008) to generalized arithmetic or quantitative thinking, giving little emphasis to the other two aspects of this way of thinking, which are functional thinking and modeling.

Also consistent with the ideas of Lins (1992) about algebraic thinking, the prevalence of the action mentioned in the previous paragraph illuminates only one of the aspects of algebraic thinking, namely: analytical thinking. The other aspects, which, for this author, is arithmetic thinking and internal thinking are not sufficiently considered.

Other actions linked to algebraic thinking that are significantly present in the conceptions evidenced by participants are: analyzing relationships between quantities, detecting structure, studying change, conjecturing, generalizing, modeling, justifying, proving and predicting (directly related to the modeling aspect, in the sense of Kaput (2008)); using a variety of representations (related to gestural, natural, pictorial and symbolic languages) that allow dealing with quantitative situations in a relational way (predominantly linked to the generalized arithmetic aspect or quantitative thinking, according to Kaput (2008)); and, in line with the 3UV Model (Ursini *et al.*, 2005), differentiating the uses of variables - unknown, functional relationship, generic number and parameter.

Less frequent, being present in only two of the 13 conceptions identified, are two actions related to algebraic thinking that we consider essential. Producing meaning for algebra and algebraic objects; one of these actions is explicitly present only in conception C8, but constitutes the essence of algebraic thinking, as Lins (1992) points out. The other action that is not frequent in their conceptions – being explicit only in C3 – is perceiving regularities in arithmetic operations. The low dissemination of this action in the teachers' conceptions about algebraic thinking may make it difficult for arithmetic to be taught since basic education, from a perspective already related to algebra and its structures, such as, for example, the properties of arithmetic operations. As Cardoso (2010, p. 130) points out:

> [...] the approach to Algebra in the early years of basic education should be based on a view of Arithmetic as part of Algebra in which arithmetic facts are seen as instances of more general ideas and in the exploration and generalization of patterns that allow functional interpretations.

Two extremely relevant actions for the development and mobilization of algebraic thinking were not evidenced by research participants in their conceptions regarding this way of thinking. The first of these is perceiving the relationships of variations and covariations. This action is directly linked to a fundamental aspect of algebra, which is thinking functionally, the seeds of which can be explored since basic education, culminating with the idea of function. It also has direct implications for the development and mobilization of proportional and variational thinking.

The other action not made explicit in the teachers' conceptions is also crucial for the adequate introjection of algebraic thinking, which involves understanding the different roles played by the equal sign (operator, indicator of equivalence). This action is related to an epistemological rupture between arithmetic and algebra, since the equal sign changes its role in relation to how it was commonly worked in arithmetic – as an operator – and starts to be explored in algebra – as an indicator of equivalence, as previously presented from the perspective of Lima; Bianchini and Lima (2023). A teacher who does not perceive the centrality of understanding the different roles of the equal sign for the development of algebraic thinking may not explore, in an appropriate way, the teaching of algebra concomitantly with that of arithmetic, proposing, even in the early years of basic education, situations in which students may be confronted with concepts and notions related to algebra, such as, for example, the issue of the status of the equal sign.

It was observed that, among the conceptions about algebraic thinking expressed by participants, there are seven that, as previously mentioned, are not explicitly related to any action linked to the algebraic way of thinking, which are C4, C5, C7, C9, C10, C11 and C12. They reveal a greater lack of clarity, on the part of the participants, about what algebraic thinking is, since they are associated with incomplete and reductionist views of algebra, the way of thinking inherent to it and to mathematics itself.

But, in the view of the research participants, what would be the themes directly involved in studies on algebraic thinking? Making these themes explicit is our aim in the next section.

Themes that, in the view of the research participants, are related to algebraic thinking

In this article, in addition to aiming to highlight the participants' conceptions regarding what they understand as algebraic thinking, we sought to identify which, in their understanding, are the themes involved in studies related to this way of thinking. To achieve this objective, we analyzed the answers given by participants to question (b) of the aforementioned initial questionnaire, namely: according to your understanding, which themes would be involved in studies on algebraic thinking? Based on the participants' answers to this question, we elaborated Table 6, in which we present the mentioned themes and the number of occurrence of each one of them.

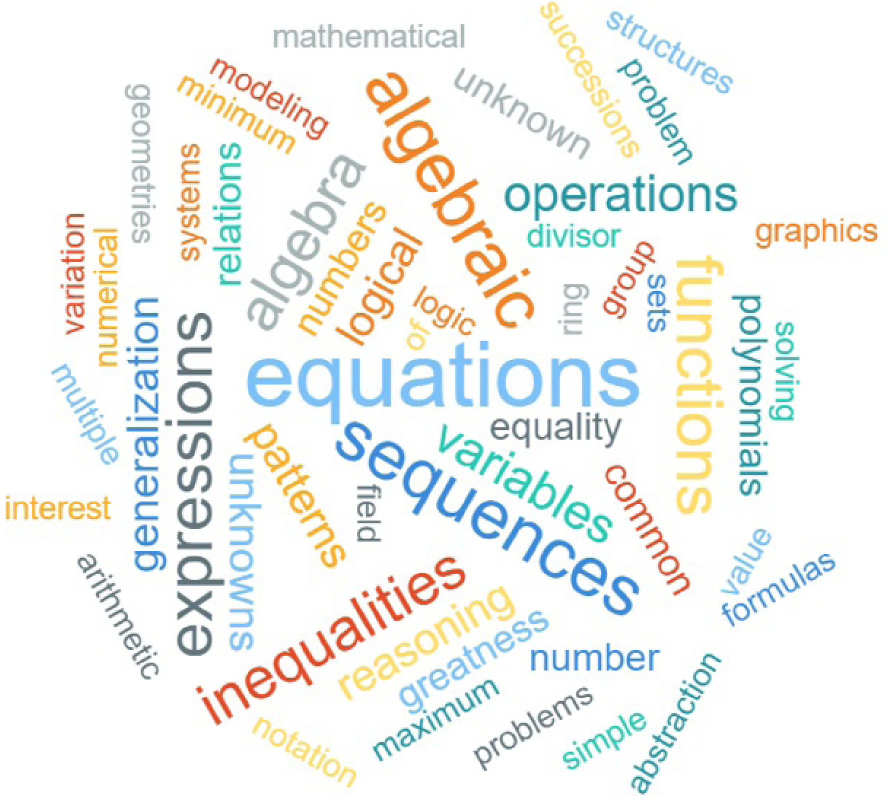
Table 6 – Themes that, according to participants, are linked to algebraic thinking

<table>
<tr><th>Themes</th><th>Number of occurrences</th><th>Themes</th><th>Number of occurrences</th></tr>
<tr><td>Abstraction</td><td>1</td><td>Unknowns</td><td>5</td></tr>
<tr><td>Algebra</td><td>5</td><td>Inequalities</td><td>4</td></tr>
<tr><td>Ring</td><td>1</td><td>Interest</td><td>1</td></tr>
<tr><td>Arithmetic</td><td>1</td><td>Logic</td><td>4</td></tr>
<tr><td>Sets</td><td>1</td><td>GCD</td><td>1</td></tr>
<tr><td>Body</td><td>1</td><td>MCM</td><td>1</td></tr>
<tr><td>Inequalities</td><td>1</td><td>Modeling</td><td>1</td></tr>
<tr><td>Equations</td><td>9</td><td>Notation</td><td>1</td></tr>
<tr><td>Structures</td><td>1</td><td>Numbers</td><td>2</td></tr>
<tr><td>Expressions</td><td>6</td><td>Operations</td><td>4</td></tr>
<tr><td>Formulas</td><td>1</td><td>Standards</td><td>3</td></tr>
<tr><td>Functions</td><td>6</td><td>Polynomials</td><td>2</td></tr>
<tr><td>Generalization</td><td>3</td><td>Problems</td><td>2</td></tr>
<tr><td>Geometries</td><td>1</td><td>Relationships</td><td>2</td></tr>
<tr><td>Graphs</td><td>1</td><td>Sequences</td><td>8</td></tr>
<tr><td>Quantities</td><td>2</td><td>Systems</td><td>1</td></tr>
<tr><td>Group</td><td>1</td><td>Value</td><td>1</td></tr>
<tr><td rowspan="2">Inequalities</td><td rowspan="2">2</td><td>Variation</td><td>1</td></tr>
<tr><td>Variables</td><td>4</td></tr>
</table>

Source: Research data.

The theme most associated by participants with algebraic thinking is equations (with 9 occurrences), followed by sequences (with 8 occurrences), expressions and functions (both with 6 occurrences), algebra and unknowns (both with 5 occurrences). Inequalities, logic, operations and variables appear with 4 occurrences. The other themes are present with 3 or fewer occurrences. The perceptions provided by Table 6 regarding the themes that participants associate with algebraic thinking can be better illustrated by means of a word cloud, such as that shown in Figure 5.

Figure 5 – Themes related to the development of algebraic thinking



Source: https://makewordcloud.com/pt/word-cloud-maker - prepared by the authors.

We highlight the fact that the arithmetic theme has occurrence 1. Once again, we highlight the disconnection between arithmetic and algebra and, consequently, between arithmetic and algebraic thinking, evidenced by research participants, an aspect also highlighted in the previous section when we discussed the low prevalence, in the teachers' conceptions, of the action of perceiving regularities in arithmetic operations.

The fact that participants did not highlight, as previously indicated in Table 5, the action of perceiving the relations of variations and covariations in their conceptions of algebraic thinking, in our view, explains the low incidence of mention of themes of variation, relationships and modeling, since it is precisely the perception of relationships of variations and covariations that makes it possible, through an algebraic approach, to model a problem from different areas. We were also surprised by the low incidence of themes of notation, abstraction and generalization, which are essential for the development of algebraic thinking. Now we present the considerations that can be inferred from these analyses.

Concluding remarks

The aim of this article was to analyze the conceptions of 36 mathematics teachers, participants of different ages and backgrounds, about algebraic thinking and what themes these participants associate with this way of thinking. These teachers participated in three synchronous online workshops, dealing with algebraic thinking and its development, which took place in the second half of 2023.

Data produced allowed us to classify the conceptions of these participants into 13 categories, which although, in general, are directly related to the actions linked to algebraic thinking, leave aside some central aspects linked to this type of thinking. In particular, we highlight the lack of mention of actions of perceiving the relationships of variations and covariations and of understanding the different roles played by the equal sign (operator, indicator of equivalence) and the reduced reference to actions of producing meaning for algebra and for algebraic objects and of perceiving regularities in arithmetic operations.

As highlighted throughout the analyses presented, the weak connection or even the total absence of articulation between algebraic thinking and such actions can have serious didactic implications. Specifically, they can lead to an emphasis on the separation between arithmetic and algebra, attributing to the former the character of a prerequisite for the latter, instead of an approach to arithmetic already focusing on the development of algebraic thinking.

They can also compromise the development, from the early years of basic education, of notions related to the concept of function, a mathematical object essential for the study and modeling of phenomena in different areas of knowledge. Specifically in relation to the concept of function, the lack of perception of the relationships of variations and covariations can hinder the dynamic understanding of the concept, crystallizing a static notion for the idea of function.

In turn, the lack of articulation between what characterizes algebraic thinking and the understanding of the different roles played by the equal sign (operator, indicator of equivalence) can have as a didactic consequence the failure to adequately explore situations that enable the understanding of the notion of equivalence, a fundamental idea in the entire development of algebra and, particularly, indispensable in solving equations and inequalities, with meaning and understanding and not merely through technical artifices.

The issue of meaning, highlighted at the end of the previous paragraph, is in no way associated only with the processes of solving equations and inequalities. It should be the focus of the teaching and learning processes of algebra, which, according to considerations of different authors mentioned in this article, occurs precisely through the development of algebraic thinking. However, if, out of a group of 36 teachers, only one of the conceptions – among the 13 identified – stated that the production of meaning for algebra and algebraic objects was explicit as an action linked to the algebraic way of thinking, once again, there is a risk that this production of meanings will not be successful if this element is no longer discussed with the teachers who will mediate the students' learning process.

The future steps of this research will be to understand the knowledge of these teachers when developing activities aimed at developing algebraic thinking and the results, in terms of expanding their conceptions about this way of thinking, of their participation in the aforementioned workshops[5].

It is noteworthy that, since this is a qualitative research, there is no intention whatsoever to establish generalizations in relation to the results obtained, beyond the group considered. We are aware that the sample is peculiar and not comprehensive enough to allow extrapolating the results, which, despite this, shed light on important aspects for reflection by teachers and mathematics educators.

The fact that the number of meetings held with participants and the time of interaction between them may not have been sufficient to capture their conceptions about algebraic thinking with total precision is a study limitation. Therefore, it could be inferred from data obtained: evidence about such conceptions, which may have been made explicit based on the elements that the research participants frequently use in their teaching practices. It is likely – and future research may or may not confirm this perception – that changes in the variables considered in this investigation may also significantly alter the results obtained.

Finally, another limitation concerns the instrument used to produce data, since a survey using a form is not always able to explain, in a global manner, everything that a teacher has to report about a given topic. In this sense, it is possible that the answers provided by participants are the translations of what they were thinking at the specific moment in which they answered the questionnaire. It cannot be assumed that, beyond what the teachers externalized, there is nothing else or that they do not effectively explore in their practice something that, when they answered the questionnaire, they did not evidence.

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