Developing Flexibility in Mental Calculation

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ABSTRACT – Developing Flexibility in Mental Calculation. Flexibility in performing mental calculations has become an important focus for mathematics educators, with research surging over the past two decades. Contemporary research results lends strong support for the development of mental flexibility in elementary classrooms. This report focuses on three interrelated themes: (1) a model of mental calculation that allows the study of flexible solution processes; (2) a review of definitions of flexibility and summary of reported research; and (3) an examination of the prerequisites for promoting mental flexibility. The approach termed *Zahlenblickschulung* is introduced with examples of activities that encourage students to sort numerical problems and reason about the sorting.
Keywords: **Flexibility. Mental Calculation. Approach to Foster Flexibility. Elementary Classroom.**

RESUMO – Desenvolvendo Flexibilidade no Cálculo Mental. A flexibilidade na realização de cálculos mentais tornou-se um foco importante para os educadores em matemática, com pesquisas surgindo nas últimas duas décadas. Resultados de pesquisas contemporâneas oferecem sólido suporte para o desenvolvimento da flexibilidade mental no ensino fundamental. Este estudo concentra-se em três temas inter-relacionados: (1) um modelo de cálculo mental que permite o estudo de processos flexíveis de resolução; (2) uma revisão das definições de flexibilidade e uma síntese da pesquisa reportada; e (3) um exame dos pré-requisitos para promover a flexibilidade mental. A abordagem denominada *Zahlenblickschulung* é apresentada com exemplos de atividades que incentivam os alunos a classificar operações numéricas e a raciocinar sobre a classificação.
Palavras-chave: **Flexibilidade. Cálculo Mental. Abordagem para promover a flexibilidade. Ensino fundamental.**

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Introduction

> The emphasis in teaching arithmetic has changed from preparation of disciplined human calculators to developing children's abilities as flexible problem solvers (Anghileri, 2001, p. 79).

In recent decades important research in mathematics education has been aimed at identifying and understanding students' techniques for performing mental addition and subtraction. In this context, students' ability to solve multi-digit arithmetic problems, without using paper and pencil computing algorithms, has come under increasing scrutiny among researchers (see e.g., Blöte; Klein; Beishuizen, 2000; Heirdsfield; Cooper, 2004; Rathgeb-Schnierer, 2006; Threlfall, 2009).

At the beginning of the twenty-first century, the National Council of Teachers of Mathematics – NCTM – (2000) argued that students should be able to use a wide variety of problem-solving strategies and that they should be able to adjust familiar strategies as well as invent new ones. Critical to the effective use of mental strategies is cognitive flexibility, an attitude of mind that is both adaptive and agile. In this context, the past decade has seen significant gains in our understanding of mental processes that contribute to or make up mental flexibility (e.g., Benz, 2007; Gruessing; Schwabe; Heinze; Lipowsky, 2013; Rathgeb-Schnierer; Green, 2013; Selter, 2009; Threlfall, 2009). In addition, new approaches have been invented for conducting empirical research on mentally flexible strategies and procedures (e.g., Rathgeb-Schnierer; Green, 2013; Threlfall, 2009; Torbeyns; De Smedt; Ghesquière; Verschaffel, 2009).

Polya's (1952) remarkable treatise on problem solving presaged much of the current state of mathematical pedagogy, which presumes rather than proves, a natural superiority of cognitive flexibility vis-à-vis rigidity where both attitudes of mind can lead to a logical, correct, and efficient solution. Only the flexible mind, however, can imbue simple problem solving with deep meanings and connections that transcend mechanical step-by-step algorithmic computations. On the other hand, it is even more advantageous to have a well-developed, flexible attitude when confronting real-world problems that often tend to be more complex because they involve novelty, ill-structured and richly-structured circumstances, and multi-dimensional situations. In this paper, we examine processes of mental calculation, survey published definitions of flexibility with their commonalities and differences, summarize reported research, and suggest pedagogical strategies for promoting mental flexibility in primary school.

Mental Calculation

Mental calculation means solving arithmetic problems (addition, subtraction, multiplication, and division) mentally without using a standard written procedure. Standard written procedures focus on calculation with single units (the ones, the tens, the hundreds, etc.); they require knowledge of basic facts and procedural knowledge of the appropriate standard algorithm. In contrast, mental calculation, especially with multi-digit numbers, is more complex because it requires one to deal with entire numbers (Krauthausen, 1993). In such a context, a deep understanding of numbers, operations, and their relations is required, in addition to knowledge of basic facts and fact families (Heirdsfield; Cooper, 2004; Threlfall, 2002).

A number of mental calculation strategies for multi-digit addition and subtraction have been formally categorized using different names and different numbers of categories (see e.g. Carpenter; Franke; Jacobs; Fennema, 1997; Fuson; Wearne; Heibert; Murray; Human; Oliver; Carpenter; Fennema, 1997; Klein; Beishuizen; Treffers, 1998; Thompson, 1999; Threlfall, 2002). The basic criteria for classifying mental calculation strategies are similar among all authors and concern basic elements of splitting up numbers into tens and ones (both numbers or only one number), rounding, and compensating. A typical approach can be found in Selter (2000), who described six main strategies for mental addition and subtraction (see Table 1).

- Jump strategy: The first step of this strategy is characterized by keeping together the first addend, the minuend, and splitting up the second addend, the subtrahend. In the second step the split numbers are successively added or subtracted.
- Split strategy: This strategy is characterized by splitting up both numbers in the problem and adding or subtracting the units separately. Regarding addition, this strategy provides an effective simplification of a complex multi-digit problem. Regarding subtraction, it is also a simplification, but only with problems that do not require regrouping.
- Mix of split and jump: This strategy represents a mixture of both strategies described above.
- Compensation strategy: In this strategy one number gets rounded to modify the problem to an easier one. Subsequently, the result gets compensated by the rounding factor.
- Simplifying strategy: The problem is modified without changing the result. For addition, this means changing both addends in an opposing way. In a subtraction problem, the minuend and subtrahend are modified in the same way.
- Indirect addition: This subtraction strategy invokes adding up from the subtrahend to the minuend. The strategy is very effective, especially if minuend and subtrahend are close together (e.g., 72-69, 69+3=72).

Table 1 – Strategies for Mental Addition and Subtraction

	Addition 56 + 38	Subtraction 91 – 46
Jump strategy	56 + 30; 86 + 8	91 – 40; 51 – 6
Split strategy	50 + 30; 6 + 8; 80 + 14	90 – 40; 1 – 6
Mix of split and jump	50 + 30; 86 + 8	90 – 40; 51 – 6
Compensation strategy	56 + 38; 56 + 40; 96 - 2	91 – 50; 41 + 4
Simplifying strategy	56 + 38; 54 + 40	90 – 45
Indirect addition		46 + ___ = 91

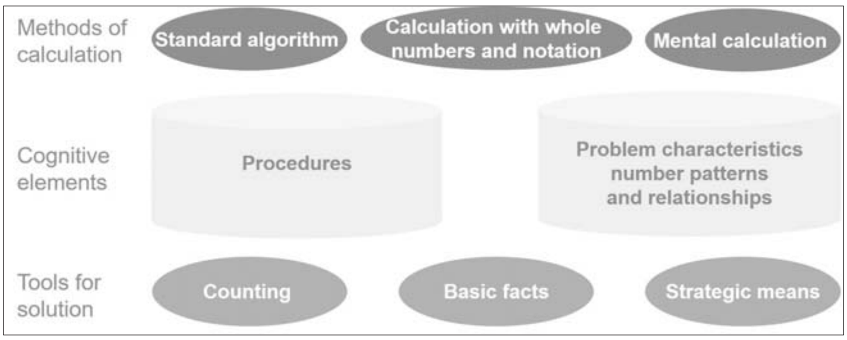
Source: Authors (2018).

The six strategies identified in Table 1 represent idealized types and do not reflect actual students' various and individual approaches for solving addition and subtraction problems (Threlfall, 2002). The strategies can be helpful in analyzing students' solutions in general but are not sufficient to give a deep insight into the mental processes of solving a problem.

A Model of Mental Calculation Processes

For analyzing and describing processes of mental calculation in detail, Rathgeb-Schnierer (2011) proposed a model with distinct but interrelated domains. Each domain is characterized by a different function and a different degree of explication. Figure 1 identifies these elements as *methods of calculation, cognitive elements, and tools for solution.*

Figure 1 – Domains of Calculation Process

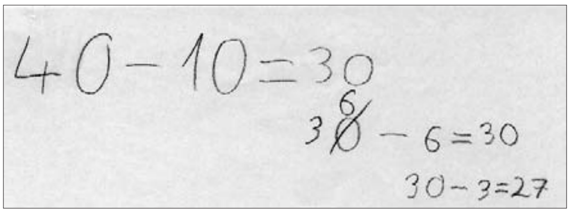


Source: Rathgeb-Schnierer and Green (2013, p. 354).

Methods of Calculation

Three different methods of calculation can be used for solving a given problem: (1) the standard computing algorithm, (2) mental calculation with whole numbers and notation (e. g., partial sums or methods with students' idiosyncratic notations), and (3) mental calculation (Selter, 2000). Although methods of calculation can sometimes be directly observed in the solution process (see Figure 2), they don't reveal detailed information about the process itself. Each of the three methods describes only one way a solution can be found but not the underlying pathways that lead to how an answer is exactly determined.

Figure 2 – Michael Solves the Problem 46-19



Source: Rathgeb-Schnierer (2006, p. 230).

Michael's notation in Figure 2 relies on the method *calculation with whole numbers and notation* and uses the strategy *mix of split and jump* (see Table 1). To obtain the solution for 46-19, he conducted several steps of computation. First, Michael split up the numbers of the problem into tens and ones (40-10) and obtained the result 30. Then he proceeded to recombine the tens and ones (30+6) and subtracted 9 from 36 in two steps (36-6=30 and 30-3=27). Note that the actual process of subtracting the numbers in each solution step cannot be assessed by looking only at the method used. There are, for instance, many possible tools for solution that Michael could have used to find the answer for 40 minus 10, such as counting or deploying adaptive strategic means combined with basic facts (deriving the answer from the analog basic fact 4-1 that he knows by heart). In short, obtaining a problem solution by itself does not shed light on the mechanism(s) used to achieve that solution.

Cognitive Elements

Cognitive elements are defined as specific mental actions that sustain a solution process (Rathgeb-Schnierer; Green, 2013). These can be learned *procedures* (such as computing algorithms) or recognition of *number characteristics* (such as number patterns and relations). Note that Michael's solution process shown in Figure 2 does not exhibit any underlying cognitive elements. One cannot tell from the notation whether Michael used a procedural solution, conducted mentally but mechanically, or whether he recognized the number characteristics and based his solution on number knowledge and number sense. In order to accurately judge his abilities in mental calculation, it would be crucial to know which cognitive elements were utilized to obtain his solution.

Tools for Solution

To find the answer to a problem, specific tools for solution may be used and combined in context. Those tools for solution may be counting, referring to basic facts, or employing adaptive strategic means (Rathgeb-Schnierer; Green, 2013). Strategic means "[...] are not holistic strategies or cognitive menus that complete a solution path; rather, they are distinct devices that can be combined in flexible ways to modify complex problems to make them easier" (Rathgeb-Schnierer; Green, 2013, p. 354). Strategic means are comprised of decomposing and composing (46-19=46-10-6-3), transforming a problem (46-19=47-20), deriving the solution from a known problem (if 46-20=26, then the answer to 46-19 must be 27, since 19 is one less than 20), and using decade analogies (40-20 is the tens equivalent of 4-2) (see Threlfall, 2002).

Analyzing Students' Solution Processes

Elements of all three domains can be combined when students solve a problem mentally. In such a context, the model can be used to analyze solution processes and identify students' competencies in mental calculation. Two examples from Rathgeb-Schnierer and Rechtsteiner (2018) help illustrate the elements. Simone (S) and Michael (M) are German students at the end of second grade. At that time, German students are familiar with addition and subtraction of two-digit numbers, but only based on calculation with whole numbers. Standard written algorithms are introduced in Germen schools in third grade. Simone and Michael used different strategies to solve the problem 71-36 at the end of second grade. The model introduced above offers an opportunity to analyze their solution processes:

Simone (S) solves 71-36

> S: *I take away one from 71, and then 70 minus 36 is four- (…) 40 - (...) - 70 minus 30 is 40 minus 6 is 34 plus 1 is 35.*

Michael (M) solves 71-36

> M: *Okay here, I do now 70 minus 35, and then I know the answer immediately; it is 35.*
> I: *And why do you know the answer immediately?*
> M: *Because then I have (points at the number 36) - 35, and 35 is half of 70, and then here* (points at the number 71) *I have 70.*

Methods of Calculation

Regarding this domain the analysis is straightforward. Neither child used paper and pencil, nor did they make any notation of their solution procedure. They both relied on different forms of mental calculation.

Tools for Solution

Tools for solutions are not directly visible; therefore, they are harder to capture. Based on Simone's statements, one can assume that she modified the problem to a related one, 70-36. Since she was not able to solve the revised problem immediately, she used the strategic means of decomposing and found the answer to 70-30. Whether she got this answer by deriving it from the analog problem 7-3 that she might know by heart is not obvious. But, since Simone found the answer very quickly, it can be assumed that she did not rely on counting. She proceeded by subtracting 6 from 40 and finally added 1 to compensate the minuend. Michael exhibited different tools for solution. He began by simplifying the original problem by modifying both the minuend and subtrahend in the same way (subtracting one from each number). After this modification, the answer to the problem became clear for Michael because he knew that 70-35 is a double-half fact.

Cognitive Elements

Did Simone and Michael rely on learned computation procedures or on recognized number patterns and problem characteristics? Michael's explanation clearly shows that his solution process is linked to this specific problem and its characteristics. It is plausible that he recognized 71 is a number close to 70 and combined that modification with his knowledge of the double-half fact. Regarding Simone, it is not so easy to reconstruct the cognitive elements that sustain her solution because she did not describe her reasoning. Without asking her why she modified the problem, one cannot judge if she was using a learned procedure or specific problem characteristics. Still, both examples illustrate how the model allows a detailed analysis of students' solution processes and a framework for understanding their mental calculations.

Flexibility in Mental Calculation

Defining Mental Flexibility

Research in the field of flexible mental calculation reflects not only different interests and aims, but also different ideas about mental flexibility that ultimately influence both the research methods and the interpretation of data. In short, there exist in the current literature multiple and inconsistent definitions of mental flexibility, and these in turn result in vastly different operational definitions (Star; Newton, 2009). For example, Star and Newton (2009, p. 558) define flexibility as "[...] knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem". In a different vein, Verschaffel, Luwel, Torbeyns, and Van Dooren (2009) distinguish between *flexibility*, to describe the use of multiple strategies, and *adaptivity*, for the actual selection of appropriate strategy choices. In this context, Selter (2009) extends the idea of *adaptivity* to include the idea of creatively developing and selecting for use an *appropriate strategy*. Finally, another group of researchers contends that flexibility consists of "[...] choosing among different strategies simply on the basis of the characteristics [...] wherein strategy flexibility is conceived as selecting the strategy that brings the child most quickly to an accurate answer to the problem" (Torbeyns; De Smedt; Ghesquière; Verschaffel, 2009, p. 583).

There are inherent problems with these definitions of flexibility. Star and Newton (2009) have shown, for example, that even mathematics education experts may fail to utilize the *most appropriate* solution strategy. And while time and accuracy may be important considerations, they do not necessarily imply cognitive flexibility in solution procedures. To overcome such problems, Rathgeb-Schnierer (2011) and Rathgeb-Schnierer and Green (2013) have taken a more intuitive approach to defining mental flexibility in terms of students' knowledge and use of number patterns and relationships. More specifically, they defined flexibility as a combination of strategic cognitive actions employed "[...] to match strategic means to recognized number patterns and relationships of a given problem in the context of processing a problem solution" (Rathgeb-Schnierer; Green, 2013, p. 357). Such a definition reflects attention to both solution methods as well as number patterns and relationships embedded in problem structures. In this sense, their definition incorporates the use of multiple strategies and reflects the kind of dynamic, adaptive thinking Threlfall (2002, p. 29) describes as "interaction between noticing and knowledge".

There is meaningful consensus among all these definitions in the idea that flexibility in mental calculation includes at least two central features: the knowledge of different solution methods and the ability to adapt those methods to a particular problem structure. Based on various definitions, Threlfall (2009) identified two different explanatory models for flexibility in mental calculation: one that reflects the idea of conscious or unconscious strategic choice and one that reflects the idea of *zeroing in* on a solution based on number knowledge and conceptual understanding. Most recently, Rechtsteiner-Merz (2013) systematically analyzed the various notions of flexibility in the literature. She pointed out that many definitions explicitly distinguish between flexibility and adaptivity. Whereas flexibility is consensually understood in the same way, there are three different approaches to defining adaptivity and how it can be identified: (1) appropriateness of solution path and task characteristic, (2) appropriateness of correctness and speed, and (3) appropriateness of cognitive elements that sustain the solution process.

All three of these approaches to defining mental flexibility can be linked to the model proposed earlier (see Figure 1). For example, the first and second approaches focus predominantly on a single domain of the calculation process: either the domain *methods of calculation* or the domain *tools for solution*. The third approach takes the cognitive elements into account and looks at two different domains to identify the degree of flexibility in students' mental arithmetic: *tools for solution* and the *cognitive elements* that sustain the solution processes. In this approach, evidence of flexibility in mental calculation can exist only if the tools for solution are linked in a dynamic way to problem characteristics, number patterns, and relationships.

Research Findings on Mental Flexibility

Not unsurprisingly, different research definitions have led to different research methods, which in turn have produced consistent patterns of results within a broad spectrum of content:

- After students learn a standard computing algorithm, they tend to stop using previously learned strategies, even when those are more advantageous and appropriate (Selter, 2001).
- When students learn by examples, they acquire specific procedures rather than general rules, and those procedures tend to have a negative impact on the development of flexibility (Beishuizen; Klein, 1998; Heirdsfield; Cooper 2004; Schütte, 2004b).
- Learning of strategies can depend on a variety of factors, such as the target operation (Torbeyns; De Smedt; Ghesquière; Verschaffel, 2009), specific numerical or problem characteristics (Blöte; Klein; Beishuizen, 2000; Torbeyns; De Smedt; Ghesquière; Verschaffel, 2009), and students' own recognition of problem characteristics (Rathgeb-Schnierer, 2006; 2010; Rathgeb-Schnierer; Green, 2015; 2017a).
- Flexible, adaptive expertise in mental calculation is associated with: deep understanding of number relationships and arithmetic operations, knowledge of basic facts and fact families, high self-confidence, and a positive attitude towards mathematics (Heirdsfield; Cooper, 2004; Threlfall, 2002).
- The development of flexibility in mental calculation can be supported by special approaches to math education. In this regard researchers have highlighted the problem-solving approach in general (Heinze; Marschick; Lipowsky, 2009; Gruessing; Schwabe; Heinze; Lipowsky, 2013; Heinze; Arend; Gruessing; Lipowsky, 2018) combined with specific activities for fostering number sense and metacognitive competencies (Rathgeb-Schnierer, 2006; 2010; Rechtsteiner-Merz, 2013).
- Students with learning difficulties in arithmetic need special instructional approaches to develop flexibility in mental calculation (Verschaffel; Torbeyns; De Smedt; Luwel; van Doreen, 2007). They exhibit conceptual progress from a particular approach to math education (*Zahlenblickschulung*) that incorporates opportunities to discover, construct, organize, and evaluate numerical patterns and relationships (Rechtsteiner-Merz, 2013; Rechtsteiner-Merz; Rathgeb-Schnierer, 2015).
- German and American elementary students exhibit similar repertoires and patterns of cognitive flexibility with multi-digit addition and subtraction problems (Rathgeb-Schnierer; Green, 2015; 2017a).

Research reported in the field of flexible mental calculation reflects different interests and aims as well as different definitions. For mathematics educators, there are also different emphases regarding pedagogy and the promotion of flexibility in mental calculation (Selter, 2009; Threlfall, 2009). According to Threlfall (2009, p. 552), “The assumption that it is all a matter of strategic choice will lead to very different conclusions about appropriate action from the perspective that presumes the importance of conceptual understanding and thinks of some kind of calculating as ‘zeroing in’ on solutions”. Two different teaching approaches can be derived from these two distinct assumptions about flexibility in mental calculation. On the one hand, flexibility in mental calculation can be supported by direct instruction, that is, by teaching students specific strategies (in the sense of a whole solution path) and encouraging them to try out and discuss the appropriateness of single strategies in specific problem-solving contexts. On the other hand, teaching flexibility in mental calculation emphasizes the development of conceptual understanding about numbers and operations that incorporates a deep knowledge about numbers, relations between numbers, as well as strategic means. We expand on the latter approach in the following section.

Supporting Flexibility in Mental Calculation in Elementary School

How can flexibility in mental calculation be supported and nurtured in elementary school? Before answering this pedagogical question, it is crucial to reflect on the prerequisites for flexibility in mental calculation. Based on research cited earlier, four elements identified in Figure 3 may be considered to be essential for mental flexibility:

- Knowledge about numbers and operations, such as knowledge about number characteristics and number relations (e.g., cardinal and ordinal values) (Heirdsfield; Cooper, 2002; 2004; Hope, 1987; Threlfall, 2002).
- Knowledge of basic facts.
- Knowledge of strategic means that includes the ability to compose and decompose numbers, to derive solutions from a known problem, to modify problems, and to use analogies (Threlfall, 2002; Rathgeb-Schnierer, 2006).
- Recognition of number patterns, problem characteristics, and relationships. This means, in the context of processing a problem solution, that students recognize number patterns and relationships of a given problem and adapt strategic means based on this recognition (Macintyre; Forrester, 2003; Rathgeb-Schnierer, 2010; Schütte, 2004b; Threlfall, 2009).

The relevance of the fourth element has become increasingly recognized in recent years. For example, several researchers have suggested that solution methods employed by students may depend more on problem characteristics than on problem types (Blöte; Klein; Beishuizen, 2000; Rathgeb-Schnierer, 2010; Torbeyns; De Smedt; Ghesquière; Verschaffel, 2009). Furthermore, Rechtsteiner-Merz (2013) has reported that students with learning difficulties tend not to develop abilities in mental calculation and seldom achieve beyond counting. Moreover, they fail to achieve a basic level of recognizing and using number patterns and numerical relations when solving problems (Rechtsteiner-Merz; Rathgeb-Schnierer, 2015; 2017).

The notion of flexibility in mental calculation as "[...] appropriate acting [which] means to match strategic means to recognized number patterns and relationships of a given problem in the context of processing a problem solution" (Rathgeb-Schnierer; Green, 2013, p. 357), as well as the requirements for flexibility outlined above, provide evidence for mathematical pedagogy. If flexibility in mental calculation means that tools for solution are used and combined dependent on recognized number patterns and relationships, then it is important for mathematics educators to address all four elements identified above. Since relying on problem characteristics, number patterns, and relationships is the foundation for flexibility in mental arithmetic, we suggest that more importance be given to activities related to this element.

One promising development on this front has been reported by Schütte (2004b) and Rechtsteiner-Merz (2013), who developed a special approach to math education that emphasizes the recognition of problem characteristics and numerical relationships. The approach is called *Zahlenblickschulung*. *Zahlenblickschulung* is a long-term approach that extends over the entire period of elementary school, and it targets the development of number concepts and the understanding of operations and strategic means (Rechtsteiner; Rathgeb-Schnierer, 2017). The basic principles of this approach are:

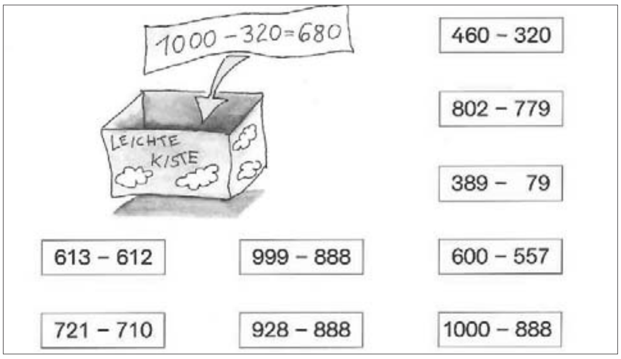
- To postpone solving problems in support of focusing on problem characteristics and relations between problems.
- To develop metacognitive competencies by posing cognitively challenging questions to provoke students' thinking and reflection.

The approach *Zahlenblickschulung* underscores and supports the development of *Zahlenblick*, which refers to "[...] the competence to recognize problem characteristics, number patterns and numerical relations immediately and to use them for solving a problem" (Rechtsteiner; Rathgeb-Schnierer, 2017, p. 2). This complex competence can be fostered by activities that encourage students to sort and arrange problems in order to recognize number patterns, problem characteristics, and relations between numbers and problems. Answers to arithmetical problems are not computed during these activities because the focus is on problem and numerical characteristics. In such situations, students have the opportunity to discover inherent structures and relations (Rechtsteiner; Rathgeb-Schnierer, 2017).

Sorting Problems: an activity for *Zahlenblickschulung*

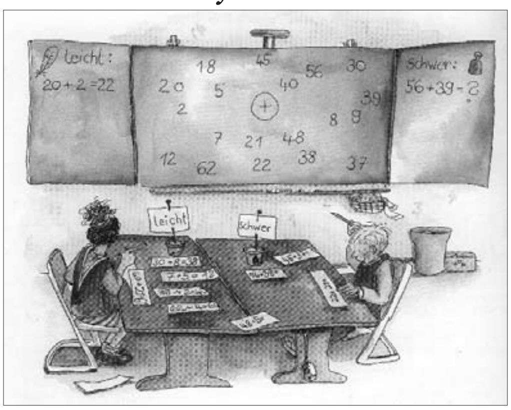
Sorting problems along with reasoning about the sort is a general type of activity that draws students' attention to problem characteristics and relationships. The activity can be used to support and develop flexibility, and it can double as a method to investigate and identify cognitive flexibility in elementary students (Rathgeb-Schnierer; Green, 2017a; 2017b). The basic idea is to encourage students to sort arithmetic problems into various categories and to discuss their reasoning about the sort. In this context, it is critical not to solve the problem before sorting, but rather to base the sorting decision on recognized problem characteristics. For the sorting activity, the teacher can employ either of two scenarios. Figure 3 illustrates the first scenario in which students sort prearranged problems into categories of *easy* and *hard*. Figure 4 illustrates the second scenario in which students, working with a specific operation and a given set of numbers, use those elements to construct problems that are both *easy* and *hard*. In both scenarios, students discuss with each other and with their teacher their reasoning behind *easy* and *hard* sorting decisions.

Figure 3 – Sorting Subtraction Problems in Categories *Easy* and *Hard*



Source: Schütte (2005, p. 54).

Figure 4 – Inventing and Sorting Addition Problems in Categories *Easy* and *Hard*



Source: Schütte (2004a, p. 28).

Depending on the grade, these activities may be adapted to the mathematics curriculum that is appropriate. Furthermore, the categories for sorting can be varied:

- *Easy problems* and *hard problems*
- *I know the problem by heart*; *I need to count the problem*; and *I know a trick to solve the problem*.
- *Problem that requires regrouping or renaming* and *problems that don't require regrouping and renaming*

The first two examples show subjective sorting categories that lead to different sorting results among students. These are also fruitful in producing actions that promote flexibility. The last example represents objective categories that should produce the same sorting results when students sort similar problems.

After sorting problems, students may be encouraged to compare their problems in each category and discuss reasons for their sorting. In this phase, cognitively challenging questions should provoke students to reason about their own and others' sortings, thereby taking problem characteristics and problem relations into account. Examples of challenging questions include:

- Why is a problem easy or hard for you? Are there special features that make problems easy or hard?
- What is the reason for assigning several different problems to the same category? Do the problems in one category have similarities?
- Are there any easy problems that can help you solve hard problems?
- Why do you know some problems by heart, and why do you need to count for other problems?
- Are there special types of problems that can be solved by the same trick? Which problems can easily be modified and made simpler?

There is growing evidence that all students benefit from the *Zahlenblickschulung* approach with regard to developing flexibility in mental calculation (Rechtsteiner-Merz, 2013; Rechtsteiner; Rathgeb-Schnierer, 2017). For students who have learning difficulties in mathematics, learning how to attend to problem characteristics and numerical relations is a critical condition for them in developing tools for solution that go beyond simple counting (Rechtsteiner; Rathgeb-Schnierer, 2017). Rathgeb-Schnierer and Green (2015; 2017a) also underscore the importance of analyzing student reasoning as a prime indicator of flexibility in mental arithmetic (as a research method) and as a fundamental approach to pedagogy in elementary classrooms.

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