OTHER THEMES

Successful Strategies of Primary School Students in Proportional Problems

Sandra Maria Pinto Magina[I]
Sintria Labres Lautert[II]
Ernani Martins dos Santos[III]
[I]Universidade Estadual de Santa Cruz (UESC), Ilhéus/BA – Brazil
[II]Universidade Federal de Pernambuco (UFPE), Recife/PE – Brazil
[III]Universidade de Pernambuco (UPE), Recife/PE – Brazil

ABSTRACT – **Successful Strategies of Primary School Students in Proportional Problems.** The article investigates the successful strategies mobilized by students in the literacy cycle when solving problems of simple proportion. 483 students from the 1st to the 3rd grade of Brazilian Elementary Education, were asked to solve six problems of proportion, and the responses of 182 students who had successful strategies were analyzed. When comparing school grades, there is an increase in hits from year to year, in the class of situations one-to-many, which cannot be said about the class for many, which remained stagnant. From the results obtained, it is possible that the multiplicative structure is being little discussed in the first three grades of elementary school.
Keywords: **Simple Proportion. Multiplicative Conceptual Field. Problem Solving.**

RESUMO – Estratégias Exitosas de Alunos dos Anos Iniciais em Situações de Proporção. O artigo investiga as estratégias bem-sucedidas, mobilizadas pelos alunos do ciclo de alfabetização ao resolverem problemas de proporção simples. Solicitou-se a 483 alunos do 1º ao 3º ano que resolvessem seis problemas de proporção, sendo analisadas as respostas de 182 alunos que tiveram estratégias bem-sucedidas. Quando comparados os anos escolares, há aumento nos acertos de ano para ano, na classe de situações um para muitos, o que não se pode afirmar sobre a classe muito para muitos, que permaneceu estagnada. Pelos resultados obtidos, é possível que a estrutura multiplicativa esteja sendo pouco discutida, nos três primeiros anos do Ensino Fundamental.
Palavras-chave: **Proporção Simples. Campo Conceitual Multiplicativo. Resolução de Problemas.**

http://dx.doi.org/10.1590/2175-623696023

Introduction

One of the great challenges for learning mathematical knowledge at school is the process of concept formation by students. In this sense, understanding how this learning takes place and the forms of reasoning mobilized by them is of great value to assist teachers in understanding this trajectory. Based on this premise, the objective of the study, presented in this article, is to investigate the successful strategies mobilized by students in the literacy cycle[1] when solving simple proportion problems.

As Vergnaud (1983; 1988; 1994; 1998) points out, the formation of a concept happens gradually and over a long period. In this sense, seeking to describe and explain the ways of thinking of children already in the literacy cycle will allow to understand if and how the concept of simple proportion is presented to children and how it can be worked by the teacher in the school context. We emphasize that this concept evokes a variety of other concepts and situations that need to be explored by teachers in classroom activities, if we want the formation and conceptual expansion of multiplicative structures to occur from this level of schooling. This is because learning certain school content means appropriating numerous concepts with which that content is related. This is one of the assumptions that underlie his Theory of Conceptual Fields, which rethinks the conditions of conceptual learning.

The Theory of Conceptual Fields

In Vergnaud's references (1983; 1988; 1994; 2009), a conceptual field can be defined as a set of situations, which brings together a variety of concepts, procedures and representations in close connection with each other. From this perspective, when studying a certain concept, we need to think about it inserted in a conceptual field. We understand that a situation, however simple it may be, involves more than one concept; in fact, it involves a network of interconnected concepts, which are necessary for the understanding and resolution of what is requested. For example, in the situation:

Ana has 12 candies and wants to share equally between her and her 3 friends. How many candies will each receive?

From the point of view of school mathematics, we can identify several concepts that are present in the above situation, such as: cardinality, grouping, two-way correspondence, distribution, division, partition, among others. They are all present in the situation and are necessary to understand the proposed situation.

When we think about mathematical problems, like the one proposed above, we notice that the situations are responsible for the meaning attributed to the concept. In the example presented, it has the meaning of division. To solve it, the student will need to make use of the operative invariants present in it. Vergnaud (1994; 1998) explains that "operative invariants" refer to the mathematical properties that are present both in the problem situation (equality, one-to-many relation, covariation relation between terms, and the distribution in equal parts and understanding about the rest in the division operation), as well as in the procedures adopted by the student when solving that class of problem. These invariants can be explained by different forms of representation: alphabetical, numerical, pictographic, iconic, among others. However, they are not always explicit to the students themselves, that is, they are not aware of the invariants present in the situation and, often, of their adopted procedure to get the answer.

Yet, when faced with a new situation, analogous to the previous experience, students will be able to use the invariants already built to analyze, understand and solve what is proposed to them. It is at this moment that the schemes - *"(...) an invariant organization of activity and behavior for a certain class of situations (...)"* - are constructed (Vergnaud, 1998, p. 168).

That is why situations (tasks / problems / activities) are at the basis of the Theory of Conceptual Fields. Starting from this, the relationship between the invariants present (explicit or implicit) and the different ways of representing them is narrowed. It is through situations that the teacher can explore students' understandings, make them potentially meaningful and direct paths for learning.

We emphasize that the Theory of Conceptual Fields provides elements for the identification of what students understand and do not understand about a certain conceptual field. In our case, it helped us in the construction of the diagnostic instrument and in the analysis of the appropriation of the concept of simple proportion by students in the literacy cycle. This is one of the concepts belonging to the multiplicative conceptual field, or, as it is often called, multiplicative structures.

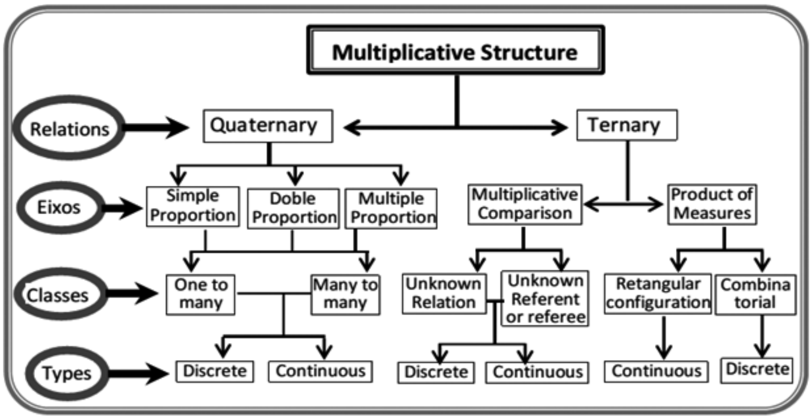
Multiplicative Structures

The multiplicative structures refer to a set of situations, which include the operations of multiplication and division, or the combination of both, which involve concepts and theorems that allow analyzing such situations. Among several concepts present in this field, we can highlight reason, proportion, fraction, divisor, multiples, rational number, linear and non-linear functions, vector space and dimensional analysis (Vergnaud, 1983; 1988).

In this direction, Gitirana et al. (2014) affirm that the work with situations involving the multiplicative field must occur during the entire basic education. This means that, in order to master multiplication and division, for example, the student needs to be able to solve different types of situations and understand the concepts that are involved in them, not just mastering the numerical calculation related to these operations.

In an attempt to organize the diversity of situations proposed by Vergnaud, which appear in different works (1983; 1988; 1994; 1998), Magina, Santos and Merlini (2014), propose a scheme, in which these concepts are classified within logic, involving relations, axes and classes and types of situations.

Figure 1 – Scheme of the Multiplicative Conceptual Field



Source: Magina, Santos and Merlini (2014).

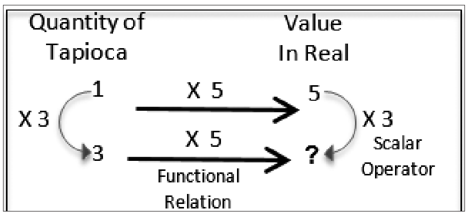
The scheme proposed by the authors points out two relations that characterize the multiplicative conceptual field: the quaternary relationship and the ternary relationship. The quaternary relation, named by Vergnaud (2009) of Isomorphism[2] of measures, presents a double relation between two or more quantities of different natures, involving four or more measures. Included in these relationships are the axes: single proportions, double proportions and multiple proportions, being discussed in this article only the simple proportions one-for-many and many-for-many, the focus of the investigation.

We clarify that the ternary relationship involves the combination of three quantities, one of which is the product of the other two, both in the numerical and dimensional plane. Part of this relationship, for example, are the combinatorial problems - Maria took 4 blouses and 2 shorts for her trip. How many different sets can she make with these pieces of clothing? - and those of multiplicative comparison - João already has 25 stickers on his album. To complete it, he will need to have three times more stickers than he already has. How many stickers does he need to complete his album?

The axis of simple proportions

This axis involves a proportionality relationship built between four measures, taken two-by-two, being two quantities of one nature and the other two of another nature (Magina; Santos; Merlini, 2014). For example: *One tapioca*[3] *costs R$ 5.00. How much will I pay when buying 3 tapiocas?* In this situation, the proportionality relationship is built between the amount of tapioca and the value in Reais. Note that, to find the unknown value, the student must establish a relational calculation between the amount of tapioca and the value in reais, as illustrated in Figure 2.

Figure 2 – Resolution scheme



Source: Elaborated byMagina, Lautert, Santos.

In the resolution scheme shown in Figure 2, we observe the existence of two operators: the scalar operator (3), which operates between measures of the same nature, and the functional relation (5), which establishes the relationship between the quantities of different natures .

When using the scalar operator (3), found in the relationship between the quantities of one measure (in this case, tapioca), we need to apply it in the relationship to be established between the quantities of the other measure (values in reais), to maintain proportionality and thus find the unknown value (the solution to the problem). The reasoning is always proportional: if the relationship between measures of one of the nature involved in the situation is established by multiplication or division with a scalar operator, that operator must be used, with the same operation, to maintain the relationship between the measures of the other nature.

We can also use, to maintain proportionality, the functional relationship (in this case the 5), found from the relationship established between the tapioca unit and the real value corresponding to that unit. In this case, we must multiply the amount of tapioca to be purchased by the value in reais corresponding to the unit, which determines the total amount to be paid (the solution to the problem). In this second possibility of resolution, the reasoning evoked is a functional relationship, in which there is a dependency between quantities of different natures, which can be operated through multiplication or division to establish the relationship between these quantities. Mathematically, we can represent the situation using the linear function $f(x) = 5x$.

The reasoning used in simple proportion situations is based on proportional knowledge (Vergnaud, 2009). It needs to be explored by the teacher in the school context, because it allows the acquisition of more sophisticated concepts, such as double and multiple proportions, which will allow the mastery of quaternary relations in the Multiplicative Conceptual Field[4].

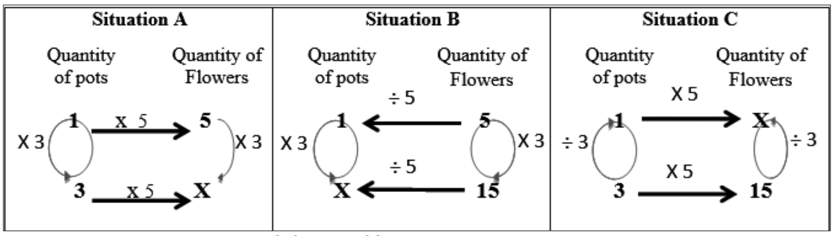
To build this proportional knowledge base, we also need to understand that proportional situations are established from two classes: *one-for-many and many-for-many,* discussed by us in several studies(Magina; Santos; Merlini, 2014; Santos, 2015; Magina; Merlini; Santos, 2016; Magina; Fonseca, 2018). In class *one-for-many*, the smallest possible relationship between the measures of the quantities involved in the situation is always explicitly stated (1 tapioca costs 5 reais).

Vergnaud (2009) also draws attention to the fact that situations of simple proportion, *one-for-many*, present variations regarding the position given to the unknown term, the quantity to be discovered for the solution of the problem. This is a central feature of quaternary relationships of this nature, having repercussions on the operation that will be employed in the search for a solution, either a multiplication or division.

Therefore, we can have the same structure to relate the quantities of different natures, but at different levels of complexity, for example (see Table 1), in which we present the resolution of the three situations, namely:

> *Situation A: João sells potted flowers in his flower shop. For each pot to be sold, he always puts 5 flowers. If he sells 3 pots, how many flowers will be sold?*
>
> *Situation B: João sells potted flowers in his flower shop. For each pot to be sold, he always puts 5 flowers. If he has 15 flowers, how many vases are needed to sell all those flowers?*
>
> *Situation C: João sells potted flowers in his flower shop. For each pot to be sold, he always puts the same number of flowers. If he has 15 flowers and 3 pots, how many flowers should João put in each pot?*

Chart 1 – Resolution scheme with different levels of difficulty



Source: Elaborated by Magina, Lautert, Santos.

Note that, in each of the situations (A, B and C), there is a quaternary relationship, which involves different quantities of nature (pots and flowers) and that this relationship is always part of the unit. However, the positioning of the unknown term (x) shows that the nature of the situation is not the same, which will require different arithmetic operations. Let us also note that, in all situations, it is possible to use the functional or scalar relationship.

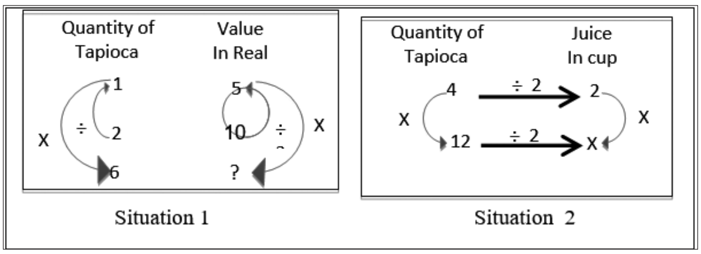
Situation A is solved by a multiplication operation, while situations (B and C) evoke division with different ways of thinking. In Situation B, there is several flowers in each pot (pre-established quota), so the amount of pots needed to place the number of flowers that João has in his flower shop should be arrived at. This situation is called quota sharing. It is important to emphasize that, often, students solve it thinking about repeated addition (adding the group of parts - in this case, the five flowers - until reaching the whole - in this case, the total of 15 flowers) Finally, in Situation C, the total amount of flowers is given, which must be distributed equally in three pots, and the size of the parts (how many flowers will be in each pot) must be found, this being called division by partition.

On the other hand, for situations of proportion established from the *many-to-many* class, the relationship between the unit value of one of the quantities in relation to the other quantity is not explicit. Regarding the set of natural numbers, in some situations, it is possible to determine this relationship, but in others it is not. Below, we will present two situations involving these two cases:

> *Situation 1: At Mr. Manoel's stall, I buy 2 tapiocas with 10 reais. To buy 6 tapioca, how much will I spend?*
>
> *Situation 2: Mr. Manoel's stall has a promotion: For every 4 tapioca purchases, the customer gets 2 glasses of juice for free. Paula and her friends bought 12 tapiocas; how many glasses of juice did they get for free?*

We observed that, in neither of the two situations above, the value of the unit of measurement of any of the quantities is explained. However, in Situation 1, it is possible to calculate it (if 2 tapiocas costs 10 reais, then 1 tapioca costs 5 reais) and, eventually, this calculation can help the student discover the amount to be paid for the purchase of 6 tapiocas (if 1 costs 5, then 6 cost 30). On the other hand, in Situation 2, although it is possible to find the unity of one of the quantities, it makes no sense to find the *one-to-many* relationship, because the condition of the problem informs that the person will only get the 2 glasses of juice, if , and only if, they buy 4 tapiocas. Therefore, in Situation 2, there is no possibility to buy 2 tapioca and get a glass of juice, that is, it does not help to solve the situation and find a *one-to-many* relationship. Figure 3 shows possible resolution strategies for each situation.

Figure 3 – Two possible schemes to solve the problems of many-to-many (Situations 1 and 2)



Source: Elaborated by Magina, Lautert, Santos.

Note that the scheme we chose to solve Situation 1 initially sought to find the *one-to-many* relationship, this being done through the scalar operator, dividing both the number 2 and 10 by 2, thus, reaching the ratio of 1 to 5. Then multiplied both 1 and 5 by 6, thus finding the value 30 for the unknown problem. This scheme could not be used to solve situation 2. We explained that the student could also use this factor without necessarily looking for the *one-to-many* relationship, that is, solving the problem by considering the *many-to-many* relationship (2 to 10). In fact, the student could identify that 6 tapiocas are 3 x 2 (so the scalar operator of tapiocas is x 3) and apply this same factor to the value greatness (in reais). In that case, he/she would find 10 x 3 = 30 reais.

Another scheme to be used in Situation 1 would be to find the functional relationship between the quantities 2 tapiocas and 10 reais In that case, it would be: 2 x 5 = 10 (functional factor: x 5) and then apply that same factor to the 6 tapiocas. The student could also use this strategy to solve Situation 2, finding the functional relationship between the magnitudes of the situation (4 ÷ 2 = 2, that is, functional factor: x / 2) then applying it to the 12 tapiocas.

There would still be the possibility for the student to solve Situations 1 and 2 by applying the 'rule of three' scheme, in which, by the algorithmic resolution procedure in the first situation, the student would multiply 6 (tapiocas) by 10 (reais) , divide the result by 2 (tapiocas) and find 30 (reais); for the second, he would multiply 12 (tapioca) by 2 (glasses of juice), divide by 4 (tapioca) and find 6 (glasses of juice). But, in this case, we thought: what sense does it make to multiply tapioca by real, divide by tapioca and find real?! Or even multiply tapioca by glasses of juice, divide the product by tapioca and find a certain amount of juice glasses?! A question similar to ours is found in Vergnaud (1998, p. 171), when he considers that, although the rule of three can be a lawful procedure to solve a simple proportion problem, class many to many, *"this procedure is used very rarely , and most students feel that it makes no sense to multiply 40km by 36 minutes. And are they, not right?"*.

Finally, we clarify that these situations are far from exhausting problems that involve the *many-to-many* relationship. We do not deal, for example, with situations when the functional factor between the quantities goes beyond the universe of whole numbers. For example, in situation 2, the promotion could be: "For every 5-tapioca bought, the customer would get 2 glasses of juice for free".

Empirical studies and the resolution of proportion problems

In a broad way, the conceptual field of multiplicative structures and, in particular, the axis of simple proportions are addressed by several authors who seek to list the importance of conceptualization in the teaching and learning process of school mathematics, the knowledge about how teaches and how to learn the concepts involved and the formulation and structure of problem situations belonging to this field.

Kishimoto (2000) investigated the effects of proportional and metacognitive reasoning on 344 Japanese elementary school students, in solving mathematical problems written with decimal fractions. Situations involving multiplication were proposed for students of the 4th, 5th and 6th grades of Brazilian Elementary School and that addressed proportional reasoning and a metacognitive questionnaire. The study identified both reasoning as factors for solving written mathematical problems and that these reasoning are important factors for 4th grade students in their solving strategies.

Pessoa and Matos Filho (2006) analyzed the skills of 153 4th and 6th grade students in solving Multiplicative Conceptual Field problems, observing the influence of schooling time on the performance of these students, comparing their resolutions analyzed in two grades. For this, students were asked to solve, individually, seven multiplicative problems of different types. As a result, it was observed that the problems of more complex types (one-to-many ratio in the division by quotas and combination) and, probably, less worked in the classroom and in textbooks, are those with the highest percentage of relational error. The authors conclude that it is still necessary to invest in the diversification of the types of problems, in their forms of representation and in the situations presented to students.

In a study carried out with 50 students in the early grades of elementary school at a public school in the metropolitan region of Porto Alegre, Lara (2011) analyzed how these students solved two problem situations that addressed multiplication (one of correspondence *one-to-many* and another *many-to-many*, both on the axis of simple proportions). It was found that students of the 1st and 2nd grades had better performance than some of the 4th and 5th grades who already used algorithms. The results of the study reflect on the usual requirement regarding the memorization of the results of a multiplication, through the use of 'multiplication tables', which can harm students in the development of their ability to think mathematically, in which the use of estimates can be valued. and creation of multiple resolution strategies by students.Magina, Santos and Merlini (2014) investigated the performance and strategies used by 175 students in the 3rd and 5th grades of elementary school in a public school in São Paulo, in solving two situations of the simple proportion axis in the Multiplicative Conceptual Field, classifying the levels of reasoning employed by them. Centering the discussions on the relationships *one-to-many* and *many-to-many*, the results indicate a limited evolution of the students' competence when dealing with situations of this conceptual field and of the investigated axis. Analyzing only the situation that addressed the *many-to-many* relationship, there was a sharp drop in this evolution. About the strategies used by 6th grade students, it was noted that these students primarily employ multiplicative procedures, while 3rd grade students use additive procedures.

The interpretation that teachers and future teachers make of the errors of elementary school students in solving problems of multiplicative structure was the object of study by Spinillo et al. (2016). It was asked to 12 future teachers and 12 elementary school mathematics teachers, in a semi-open interview, during the presentation of six cards, each containing the statement of a problem (three of product of measures, three of isomorphism of measures), which indicate the incorrect solution of the problem that should be interpreted. Participants identified procedural, linguistic and conceptual errors, characterizing errors in measurement product problems, especially as conceptual, and errors in measurement isomorphism problems, especially as linguistic ones. The same pattern of quantitative results was found for both groups, leading the authors to the conclusion that, in the teaching of mathematics, the type of problem has a relevant role in the way of interpreting errors more than the training and experience of these teachers.

In another study, Spinillo et al (2017) investigated how elementary school teachers understand and formulate situations pertaining to the Multiplicative Conceptual Field. For this, they asked 39 teachers of all grades of this level of education and who worked in public schools to formulate mathematical situations that could be solved through multiplication and/or division. The results of the study showed that the investigated teachers understand the meaning of a multiplicative situation and formulate problems appropriately, with few statements in which information was omitted or that presented linguistic inaccuracies. It was identified that most of the situations elaborated were of the same type and involved only one step for their resolution. The little variability was verified in relation to all teachers, regardless of the grade in which they taught. The authors concluded that the investigated teachers have difficulty in formulating problem situations that involve the different relationships that comprise the multiplicative structures, and it is necessary to develop together with the elementary school teachers the ability to formulate diverse mathematical problems that encompass all the complexity of the conceptual investigated field.

Merlini and Teixeira (2018) analyzed the performance of 162 students from the 1st grade of elementary school in public schools in five different regions of Bahia and categorized their adopted resolution strategies that led to the correctness, when they resolved a situation of simple proportion, class one-to-many, whose most suitable operation is multiplication. The authors conclude that even students in the 1st grade of elementary school who have not yet had formal contact with situations of the multiplicative structure have demonstrated mathematical notions, as they use iconic representation as a resolution strategy, managing to solve situations in the multiplicative structures.

Lautert, Santos and Merlini (2018) investigated the performance and resolution procedures mobilized by 809 students from the 3rd and 5th grades of elementary school in public schools in Recife and Ilhéus, in order to solve four division problems involving simple proportions: two of *one-to-many* and two of *many-to-many* correspondence. The results show that *one-to-many* correspondence problems are easier than *many-to-many* correspondence problems, for both grades; that 3rd grade students tend to present more idiosyncratic procedures/strategies or without connection with the statement, while 5th grade students tend to perform procedures involving multiplicative reasoning.

In view of the above, the study investigates the strategies of students in the literacy cycle by settling situations of simple proportion, *one-to-many* and *many-to-many*, whose aspects have been little explored in the literature of the area.

We consider the possibility of a relationship between student performance and the fact that the multiplicative structure is usually little or not worked at school until the 4th grade and when it is, it has a connotation of repeated addition (often related to multiplication tables). However, this view is strongly supported in our experience with the training of teachers in the early years of elementary school, which we recognize as having little power of generalization. What we have seen is that the school reserves the first two school grades, and sometimes the third grade as well, for the additive structure. The multiplicative structure takes place in the 4th and 5th grades, when it is usually worked in a formal way, through multiplication tables and teaching the multiplication operation, followed by division. Thus, we believe that when investigating the strategies of 1st and 2nd grade students, we will not find canonical strategies (those taught by the school, which follows the formal rigor of Mathematics) in solving problem situations. On the other hand, among 3rd grade students, it is possible that some (few) formal strategy will appear, perhaps even mixed with informal actions.

The study

It was a descriptive study composed of a diagnostic instrument that included 13 problem situations. This instrument was diagrammed in a booklet format, occupying half an A4 sheet. Every problem situation offered space demarcated below each statement for resolution and response, and, therefore, each problem situation occupied a page. The diagnosis was applied collectively, per school year, to students from all elementary schools, leaving them to resolve individually and in writing. For the purpose of this article, we will analyze six of the 13 situations that were in this diagnosis, namely, those that dealt with the simple proportion and, still, referring only to the answers offered by students from the 1st to the 3rd grades of Elementary School, that is, those who found, at the time of application, studying literacy grades. The situations that we will analyze are illustrated in Chart 2. The application of the instrument occurred in a single session and had the collaboration of the teacher. It was up to the teacher to read the problem situations out loud, one at a time, asking the students to solve it in the spaces indicated, before moving on to the next reading.

From the sample point of view, the study involved a population of 483 students, of both sexes, attending the first three grades of elementary school in four public schools in the city of Recife, who were participants in the E-Mult project, financed by CAPES, within the scope of the *Edital Observatório da Educação*. These quantitative only enrolled students who scored at least one of the six problems found in Table 2. As of this condition, 182 students remained, being 22 from the 1st grade, 45 from the 2nd grade and 115 students from the 3rd grade.

Chart 2 – Problem situations of simple proportion proposed to students

One-to-many	Many-to-many
P1. Joana knows that, in a package, there are 6 cookies. She has 5 packages. How many cookies does Joana have?	**P4.** To make 3 costumes, 5m of fabric is required. Ana has 35m of fabric. How many fantasies can she make?
P2. A supermarket did a promotion: "Take 4 liters of juice for only 12 reais". How much will each liter of juice cost?	**P5.** Caio bought 9 boxes of juice and paid 15 reais. If he bought 3 boxes of juice, how much would he need to pay?
P3. Escola Recanto will host a party for 36 guests. At each table, there will be 4 guests. How many tables will the school need to rent?	**P6.** In a contest at Escola Saber, every 3 laps running on the court, the student scored 4 points. Alex ran 15 laps on the court. How many points did he score?

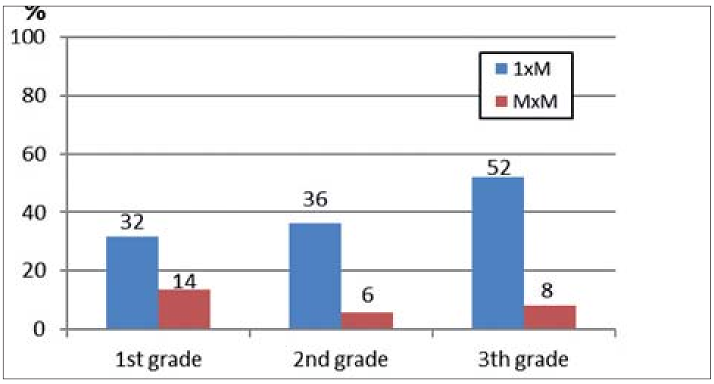
Source: Project in Rede E-Mult (Santana, 2013) / Project PEM (Magina, 2013).

Having explained the study, we will discuss the results obtained in the next section. It is important to remember that our analysis will focus only on the students' scores. In other words, our interest is to investigate successful strategies, mobilized by students from the 1st to the 3rd grade (literacy cycle) when they solve problems of simple proportion.

Analysis of results

We started by presenting the percentage of correct answers of the three groups studied (1st grade, 2nd grade and 3rd grade) in the six questions investigated, considering the general correctness and the correctness in each of the proportion classes (one-to-many and many-to-many), shown in Graph 1 below:

Graph 1 – Percentage of general and by class scores, of students of the 1st, 2nd and 3rd grades



Subtitle 1xM = *one-to-many*; MxM = *many-to-many*
Source: Magina, Lautert, Santos, based on the study data.

Graph 1 shows that when we analyze students' performances in situations *one-to-many*, in the three school grades, we find that there is a significant difference between them ($c^2_{(2)}$ = 17.064; p < 0.001). However, this difference is not confirmed between the 1st and 2nd grade ($c^2_{(1)}$ = 0.391; p =0.531), indicating that the group that presents a different behavior from the others is the 3rd grade, whose percentage of correct answers was higher than that of the previous two grades.

On the other hand, when we compare performance between school grades in *many-to-many* situations, we find that the differences are not statistically significant ($c^2_{(2)}$ = 3.584; p = 0.167). In fact, the three groups have had little success in solving this type of problem.

It is also worth noting that, when analyzing the performance of 1st grade students, according to the type of problem, we find that the average percentage of correct answers in *one-to-many* situations is almost double the average in *many-to-many* situations ($c^2_{(1)}$ = 6.212; p = 0.013). In the 2nd grade, this difference increases, reaching almost 6 times more ($c^2_{(1)}$ = 37.383; p < 0.001). Finally, in the 3rd grade, in which students' performance in situations *one-to-many* proved to be statistically better than in previous grades, this performance is almost eight times higher ($c^2_{(1)}$ = 215.182; p < 0.001). Thus, we find that the gain in learning proportionality occurs mainly in situation *one-to-many*, while in situation a *many-to-many*, it remains stagnant.

Regarding student behavior regarding the number of right problem situations, Table 1 provides an overview of all students versus the number of right situations. We clarified that students could correct six situations at most, that is, three situations in class *one-to-many* and three in *many-to-many*. We also inform that Table 1 below shows the results of all students, without separating the school grade.

Table 1 – Number of situations that students scored, according to the classes of proportions

Scores / Classes	One situation	Two situations	Three situations	Four situations	Five situations	Six Situations
The two classes (n = 182)	108	46	21	5	2	0
One-to-many (n = 161)	96	44	21	0	0	0
Many-to-many (n = 40)	36	4	0	0	0	0

Source: Magina, Lautert, Santos, based on the study data.

Note that the first line presents the answers of 182 students (regardless of school grade, or the type of class the student belongs to - *one-to-many* or *many-to-many*). So, for example, we have 108 students who scored just one situation, which may be class *one-to-many* or *many-to-many*. Likewise, and still looking at this line, we have that five (05) students scored four situations. These four situations can be three from the class of *one-to-many* and one from the *many-to-many*, or even two from the class *one-to-many* and two from the *many-to-many*. It cannot be three from the class of *many-to-many* and one from *one-to-many*, because no student has solved the three situations in the class of *many-to-many*. On the other hand, in the second and third lines of Table 1, the value of “n” does not match 182, nor is it double. In fact, it seems to indicate that there would be 201 students (161 + 40), but it is not so. What happens is that there were cases in which the student was counted twice, because he/she scored, for example, two situations in the class *one-to-many* (and then he/she is one of 46 students) and he also scored a situation in the class *many-to-many* (being part of the 36 students).

Information taken from the data presented in Table 1 refers to the level of difficulty between one and another class of situations. In fact, it is noted that while in the *one-to-many* class, 21 students scored all three proposed situations, in the *many-to-many* class, we had no students with such success. Even with the score of two problems in the class, the number of correct situations from *one to many* was 11 times higher. As we discussed earlier, we credit this result to the fact that, while in the class of *one-to-many*, the student is required only one operation, in the class of *many-to-many*, two operations are required.

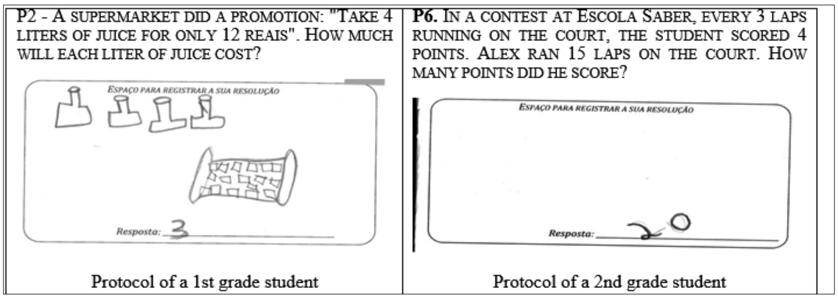
As the focus of this article is on analyzing the successful strategies used by these students, we will leave aside the questions related to the students’ skills (percentages and number of correct answers) when solving the problem situations to focus on the ones they used, achieving success in their resolutions. While 3rd grade students diversified their resolution strategies more (we identified six in all), 1st and 2nd grade students were limited to using only three strategies.

Based on the reading of the strategies used by the students when answering the diagnostic instrument, it was possible to identify six strategies. Of these, only the first three were found among 1st and 2nd grade students. From the point of view of Vergnaud (1994, 1998), the latter have less resources to deal with such situations, most likely the result of their little expansion of the multiplicative conceptual field. It is important to note that further expansion of this field by 3rd grade students does not necessarily mean school learning. Such fact may have occurred due to the cognitive development factor or, still, due to the interaction of the children in this group with multiplicative situations of everyday life.

Next, we will describe and exemplify the six strategies, which were identified with the contribution of three judges (specialists in the area of Mathematics Education), through discussions to reach a consensus on the strategy adopted by students in solving situations:

Strategy 1 (E1): misunderstood – it is characterized by the fact that it is not possible to establish a relationship between the procedure adopted by the student and the result presented by him/her, either because he/she does not make the representation clear or because he only provides a number as an answer, without any other record. Alternatively, the student makes record that did not allow us to identify the strategy he/she used to resolve the situation. We present, in Figure 4, below, two examples whose answers were classified in category E1.

Figure 4 – Examples of strategies classified as E1

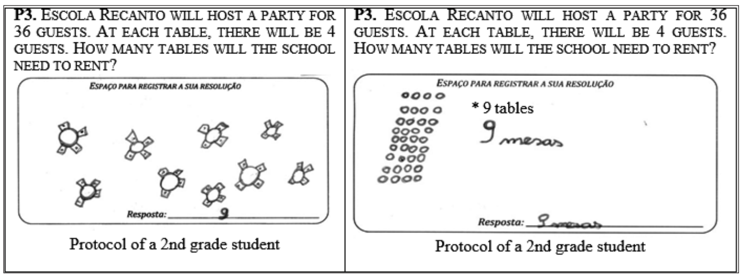


Source: Project in Rede E-Mult (Santana, 2013).

We observed that, in the two examples above, the students offered the correct answer to the situations, but they left us no clue as to how they thought to solve them. The student in the example on the left produces some drawings, which are not enough for us to understand the strategy he/she used. On the other hand, in the example on the right, the student leaves no mark on the paper beyond his/her answer, making any identification of strategy unfeasible.

Strategy 2 (E2): groupings – it is characterized by responses in which students use groupings (iconic or numerical), without however indicating whether, from that group, enumeration (counting one by one) or an operation (addition or subtraction) was used to get the answer. There is no indication that the student has made use of an operation on his/her record. Below, we present, in Figure 5, two examples of using this type of strategy.

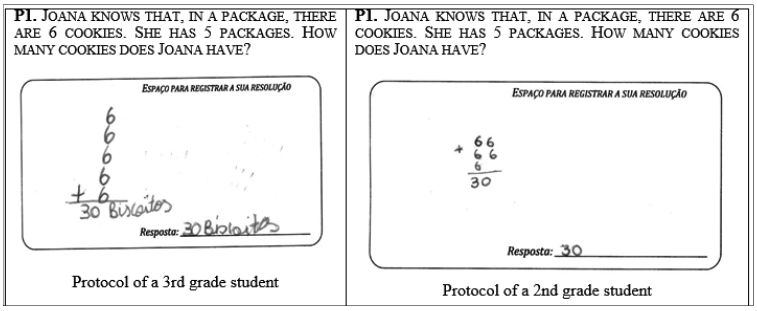
Figure 5 – Examples of strategies classified as E2



Source: Project in Rede E-Mult (Santana, 2013).

Strategy 3 (E3): repeated addition – characterized by the option for students to register an addition, numerical or iconic, of repeated plots in their action in solving the problem. We understand that, when using this type of strategy, the student demonstrates that he/she is in a transition phase between additive and multiplicative reasoning (Magina, Santos, Merlini, 2014, p. 528). "This strategy is close to multiplicative thinking, but it is anchored in additive reasoning, that is, it forms groups of the same quantity to then carry out the addition operation". Figure 6 presents two protocols that exemplify this type of strategy.

Figure 6 – Examples of strategies classified as E3

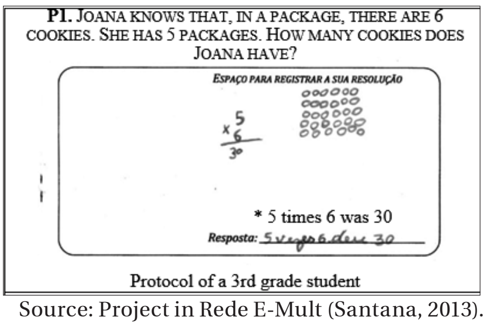


Source: Project in Rede E-Mult (Santana, 2013).

In both the example on the left and on the right, we noticed that the students repeated the number six, five times and then added that amount. While the student in the example on the left has mastery over the addition operation, from the point of view of the algorithm, the student in the example on the right does not master such an algorithm. However, note that both have the concept of addition, both repeat the numeral 6 five times and both arrive at the same correct result, in a clear demonstration that they understood that they should add the number 6 five times, satisfying the situation-problem proposed.

Strategy 4 (E4): iconic support operation – it is characterized by the act of solving the problem through a formal multiplication, but with iconic support. In this case, the student demonstrates how to arm and carry out the multiplication operation but seems to lack iconic support that presents itself as a repeated model. This support probably helped him/her to carry out this multiply operation. This type of strategy, which occurred only among 3rd grade students, points not only to the student's formal contact with the multiply operation, but also to an informal strategy search (iconic representation). Figure 7 below provides an example of this type of strategy.

Figure 7 – Example of strategy classified as E4

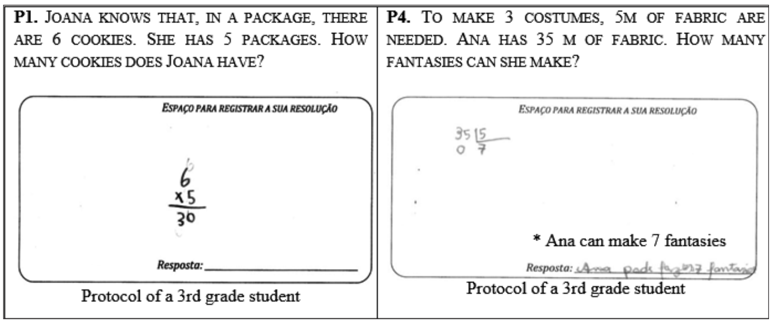


Source: Project in Rede E-Mult (Santana, 2013).

Upon examining the student's resolution, we noticed that he/she, in addition to setting and correctly calculating the account, drew, next to it, five rows of six dots. It is not clear whether the student first counted and then lined up the dots to confirm that his/her action was correct, or whether, on the contrary, he/she designed the rows of dots and, based on them, set up and carried out the account. What is clear is that the icon and the idea of repeated addition are part of its resolution, together with the account.

Strategy 5 (E5): use of multiplicative operations – it is characterized by the student's action to use an arithmetic operation (multiplication or division) in a canonical manner, to seek the solution of the problem. This strategy, like the previous one, was used only by 3rd grade students. In the sequence, we present, in Figure 8, two examples (one of multiplication and one of division).

Figure 8 – Example of strategies classified as E5

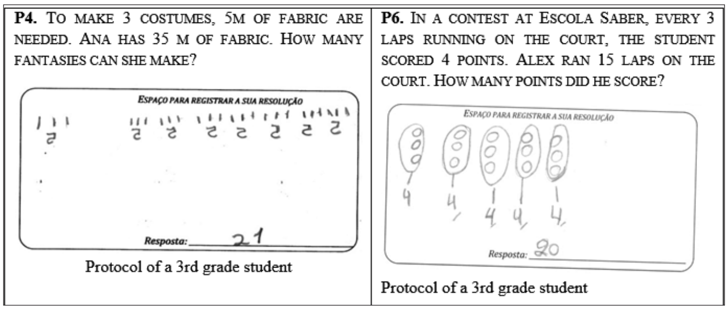


Source: Project in Rede E-Mult (Santana, 2013).

We observed that while the student in the example of Problem 1 satisfactorily solves the situation, the same does not happen in Problem 4. In fact, we note that, in problem 4, the student solves the first part of the problem (finds the value of the operator to climb between the meters, which is 7m), but does not apply this relation to costumes (3 X 7 = 21 costumes). It is important to keep in mind that while problem 1 is class *one-to-many*, which requires only one operation to find the solution, Problem 4 belongs to class *many-to-many*, which requires two operations (multiplication and division), making it a more difficult problem for students. Our statement is supported by the percentage of correct answers that these students presented in both types of problems (see Table 1, where it appears that students' performance in the problems of *one-to-many* was four times better than in those of *many-to-many*).

Strategy 6 (E6): functional with iconic and symbolic support – it is characterized by actions in which the student establishes a proportional relationship between the quantities of different natures, applying this relationship to arrive at the resolution of the problem. In Figure 9, we present two examples in which this strategy is used.

Figure 9 – Examples of strategies classified as E6



Source: Project in Rede E-Mult (Santana, 2013).

The first thing that draws our attention in the examples is that, in both, the students sought support icons (dashes or dots). The relationship they establish is between the values explicitly stated in the statements - 3 (costumes) for 5 (meters) and 3 (turns) for 4 (points). However, how did they know that they would have to repeat the '5 to 3' relationship seven times? Or likewise, how did they know that in order to know the total point that Alex scored, it was necessary to establish this '4 to 3' relationship five times?

Our hypothesis is that the student first used the covariation operative invariant (or scalar operator). In the example of meters for costumes, the student seems to have identified that 35 m is seven times more than 5m (or, still, to reach 35 from 5, it is necessary to repeat this 5 seven times). In other words, the student found the value of the scalar operator between the values of the meter variable (in case 7) and applied it to the functional relationship (between 3 meters and 5 costumes), identifying how many times this relationship would be repeated. Likewise, we can assume that, in the example on the right, the student first identifies the scalar operator (5), and this will be the number of times that the relationship between 4 and 3 will be repeated, concluding that 4 points 5 times result in 20 points.

This hypothesis may have as an action variant the idea of "complementation" in which the 5 is repeated several times until it reaches 35 (5 + 5 = 10, it did not arrive; + 5 = 15, it did not arrive ... and so on until it arrives to the desired value), and then this process is copied to the other quantity (3 seven times, reaching 21). Whether by one way or another, the fact is that the student explains the scalar factor between the values of one of the variables (between 5 and 35, since he/she repeats 7 times 5) and does the same for the other variable , repeating the 3 (represented iconically) seven times. Our student makes it clear that there is a relationship between the values of the variables (5 and 3, in the case of fabric meters, and 4 and 3, in the case of laps and points per lap).

Finally, to summarize our analysis, we find, in Table 2 below, the presentation and quantification of the types of strategies used by students of the three grades, according to the type of problem situations.

Table 2 – Relationship between the strategies used successfully and the problem situa

	1st grade (n=22)			2nd grade (n=45)			3rd grade (n=115)						Total
	E1	E2	E3	E1	E2	E3	E1	E2	E3	E4	E5	E6	
P1	3 (of 22)	6	0	9 (of 45)	16	3	17 (of 115)	30	18	7	16	0	**125** (of 182)
P2	1 (of 22)	3	1	5 (of 45)	9	0	14 (of 115)	35	1	0	5	0	**74** (of 182)
P3	7 (of 22)	1	0	5 (of 45)	0	0	25 (of 115)	7	1	0	4	0	**50** (of 182)
P4	0 (of 22)	0	1	1 (of 45)	0	0	6 (of 115)	0	0	0	0	2	**10** (of 182)
P5	7 (of 22)	0	0	5 (of 45)	0	0	11 (of 115)	0	0	0	0	0	**23** (of 182)
P6	0 (of 22)	0	0	2 (of 45)	0	0	4 (of 115)	2	0	0	0	2	**10** (of 182)
TOTAL	18 (of 132)	10	2	27 (of 270)	25	3	77 (of 690)	74	20	7	25	4	
%	14	8	1,5	10	9	1	11	11	3	1	4	0,5	

Note: n = number of students who scored at least one of the six problem situations. Subtitle: P1, P2 and P3 = Problems one-to-many P4, P5 and P6 = Problems many-to-many. E1 = misunderstood strategy; E2 = groupings strategy; E3 = repeated addition strategy; E4 = operation strategy with iconic support; E5 = strategy using multiplicative operations and E6 = functional strategy with iconic and symbolic support.
Source: Magina, Lautert, Santos, based on the study data.

The data shown in Table 2 indicate that there was a tendency for behavior in the three groups, in which the most used strategies were, by far, E1 and E2 - one that does not explain the scheme used to resolve the situation (E1) and that of grouping, through the use of icons (E2). Still, although in small numbers, we had students from the three grades using the repeated addition strategy (E3), usually with the help of icons. Such data provides us with an idea of the importance of the icon in supporting students' reasoning, as we know that multiplication problems are little or not worked in these years of schooling, especially the 1st and 2nd grades. These results reaffirm the conclusions of the study by Merlini and Teixeira (2018).

We also want to draw attention to: first, the students of the three grades had similar behaviors in the sense that it was E1, followed by E2, the most commonly used strategies; second, E1 may have been a "kick", a count or even a mental operation for these students. But what matters here is that the student did not know (or could not or did not want to) record the path he/she took to find the result. It is important to inform that, throughout the application of the test, the researchers constantly emphasized and encouraged the recording of their resolutions. The case of not knowing or not being able to register in the references to the implicit use of the invariant (Vergnaud 1994, 1998).

We also noticed that more sophisticated strategies - E4, E5 and E6 - in which the multiplication or division operation is explicitly performed - only appear among 3rd grade students and, even so, with a negligible percentage (just over 5%). This was already expected, since, as we stated earlier, it is customary for the school universe to only formally introduce the multiplicative structure in the 3rd grade, with emphasis on the 4th grade.

Finally, we would like to draw your attention to the non-canonical solutions of four 3rd grade students, who, in two of the problem situations (P4 and P6), find the solution looking for the relationship, with the help of icons, existing between the variables present in the statement. In the view of Magina et al. (Magina; Santos; Merlini, 2014; Magina; Merlini; Santos, 2016; Magina; Fonseca, 2018), the proportionality relationship is built between four measures, treated two by two.

Conclusion

Our first conclusion is that students, even without having learned the process of multiplying, can think multiplicatively. This result has found support in related studies (Magina, 2013; Magina, Santos e Merlini, 2014, Lautert; Santos, 2017, Lautert; Santos; Merlini, 2018, Merlini; Teixeira, 2018; Magina; Fonseca, 2018). We identified, however, that, even if such thinking is sufficiently elaborated, alternative strategies are already sought since the 1st grade, to lead them to success in solving the problem situation. So, although this conclusion has already been pointed out in previous studies, we bring as a novelty the identification of alternative strategies of the students, such as counting, repeated addition and, in a way, the use of different icons.

There is a clear distinction, in terms of success and use of strategies, between the two classes - *one-to-many* and *many-to-many* - in which, in the first, there is a significant jump between the behaviors of students from the 1st and 2nd grades and those of the 3rd, in favor of the latter; on the other hand, in the second, the result suffers from the floor effect. It is difficult for students to explain the invariants of the class *many-to-many*, resulting in solutions based, above all, on implicit schemes. This was true for the three school years.

Finally, we conclude that these results confirm our conjecture that the school has taught little or nothing about multiplicative structures, especially in the 1st and 2nd grades. Such a posture makes it difficult the appropriation and development of multiplicative reasoning, already present, intuitively, among some students from the 1st and 2nd grades. There is a tendency, among students, to think of the multiplicative situation under an additive look, even when they come up with the multiplication algorithm (3rd grade students)[5].

Received on August 30, 2019
Approved on October 6, 2020

Notes

1 When carrying out this study, the literacy cycle comprised the 1st, 2nd and 3rd grades of elementary school. Thus, for the purpose of this article, the term "literacy cycle" will include the grades mentioned above.

2 Isomorphism, from a mathematical point of view, is the mapping between objects, which shows a relationship between two properties or operations, preserving the structure of those properties or operations.

3 Tapioca is a typical food from Northeastern Brazil, made by roasted cassava stuffed with coconut, cheese and butter.

4 More information about proportional reasoning can be obtained (Tourniaire; Pulos, 1985, Lautert; Schliemann, 2020).

5 *Acknowledgments:* We would like to thank the *Conselho Nacional de Desenvolvimento Científico e Tecnológico* (CNPq) for the fellowship research funding. We also acknowledge the Fundação de Amparo à Pesquisa do Estado da Bahia (FAPESB) and the *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior* (CAPES) for their assistances provided in the development of the studies from which this paper come from.

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Sandra Maria Pinto Magina has a bachelor's degree in Psychology, a PhD in Mathematical Education from the University of London, a post-doctorate from the University of Lisbon. Professor in the Department of Exact Sciences and Technology, and member of the collegiate bodies of the Graduate Programs in Mathematics Education and in Education at UESC / BA.
ORCID: http://orcid.org/000-0003-0383-9744
Email: sandramagina@gmail.com

Sintria Labres Lautert holds a PhD in Cognitive Psychology (UFPE), Post-doctoral Poincaré Institute for Mathematics Education, Tufts University, Massachusetts, USA. Profa. Associate III, Department of Psychology and the Graduate Program in Cognitive Psychology UFPE.
ORCID: http://orcid.org/0000-0002-7732-0999
Email: sintrialautert@gmail.com

Ernani Martins dos Santos holds a PhD in Cognitive Psychology (UFPE); Adjunct Professor and Dean of Undergraduate Studies at the Universidade de Pernambuco (UPE), acting as Professor of the Graduate Program in Teacher Training and Interdisciplinary Practices.
ORCID: http://orcid.org/0000-0002-3824-986X
E-mail: ernani.santos@upe.br

Editor-in-charge: Beatriz Vargas Dorneles